The Geometry of Möbius Transformations

John Olsen

University of Rochester

Spring 2010

Contents

1 Introduction

The purpose of these notes is to explore some basic properties of Möbius transformations (linear fractional transformations) which are one-to-one, onto and conformal (angle preserving) maps of the socalled extended complex plane. We will develop the basic properties of these maps and classify the one-to-one and onto conformal maps of the unit disk and the upper half plane using the symmetry principle. The one-to-one, onto and conformal maps of the extended complex plane form a group denoted $PSL_2(\mathbb{C})$. We will study the conjugacy classes of this group and find an explicit invariant that determines the conjugacy class of a given map. We finish with a classification of the finite subgroups of $PSL_2(\mathbb{C})$.

The theory of Möbius Transformations is developed without any use of and only one reference to complex analysis. This point of view certainly requires more work, but I feel the effort is worth it, since it allows somebody with no knowledge of complex analysis to study the subject. The prerequisite is some basic knowledge of group theory, which is certainly met if the students have taken an undergraduate algebra course. If not, a couple of lectures at the beginning of the course where one introduces the basics of group theory should suffice.

I would like to thank John Harper and the students in the course Introduction to Geometry (MTH 250, Spring 2010) for valuable comments and proofreading.

2 Affine Transformations

Let us briefly recall a few basic properties of the complex numbers. If $z \in \mathbb{C}$, then we can write $z = r(\cos(\theta) + i\sin(\theta))$, where r is the modulus, $|z|$, of z and θ the argument, $\arg(z)$, of z. We will denote the real and imaginary part of z by $\Re(z)$ and $\Im(z)$, respectively.

We have the following five basic maps, which we will study in the following:

1. $z \mapsto cz$, $c \in \mathbb{R}$, scaling
2. $z \mapsto z + A$, $A \in \mathbb{C}$, translation
3. $z \mapsto Az$, $A = \mathrm{e}^{i\theta}$, rotation
4. $z \mapsto \overline{z}$, complex conjugation
5. $z \mapsto \frac{1}{z}$, inversion.

Definition 1. A direct affine transformation is a combination of (1), (2) and (3), i.e. a map of the form $T(z) = Az + B$.

Remark 1. Since we have

$$|T(z) - T(z')| = |(Az + B) - (Az' + B)| = |A||z - z'|,$$

we see that a direct affine transformation is an isometry if and only if $|A| = 1$.

Lemma 2.1. *The direct affine transformation $T(z) = Az + B$ is a translation if and only if $A = 1$. If $|A| = 1$, $A \neq 1$ then T is a rotation about $\frac{B}{1-A}$ by an angle* $\arg(A)$.

Proof. If $A = 1$, then $T(z) = z + B$ is a translation. If $|A| = 1$ and $A \neq 1$, we let F be a fixed point of T, i.e. a point where $T(F) = F$. Then we have $F = AF + B$, which implies $F = \frac{B}{1-A}$. Now

$$\begin{aligned} T(z) - F &= \frac{Az - A^2 z + B - BA - B}{1 - A} \\ &= \frac{Az - A^2 z - BA}{1 - A} \\ &= \frac{Az(1 - A) - B}{1 - A} \\ &= A(z - F). \end{aligned}$$

This shows that if $|A| = 1$ and $A \neq 1$ then T is a rotation about F by an angle $\arg(A)$. $\square$

Proposition 2.1. *The set of direct affine transformations form a group under composition, which is denoted by $Aff(\mathbb{C})$.*

Proof. The identity map $I(z) = z$ is the unit in the group $Aff(\mathbb{C})$. Since composition of functions is always associative, we only need to check that the set is closed and that inverses exist. Let $T(z) = Az + B$ and $S(z) = A'z + B'$, then we have

$$S \circ T = A'(Az + B) + B' = A'Az + A'B + B',$$

which is again an affine transformation.

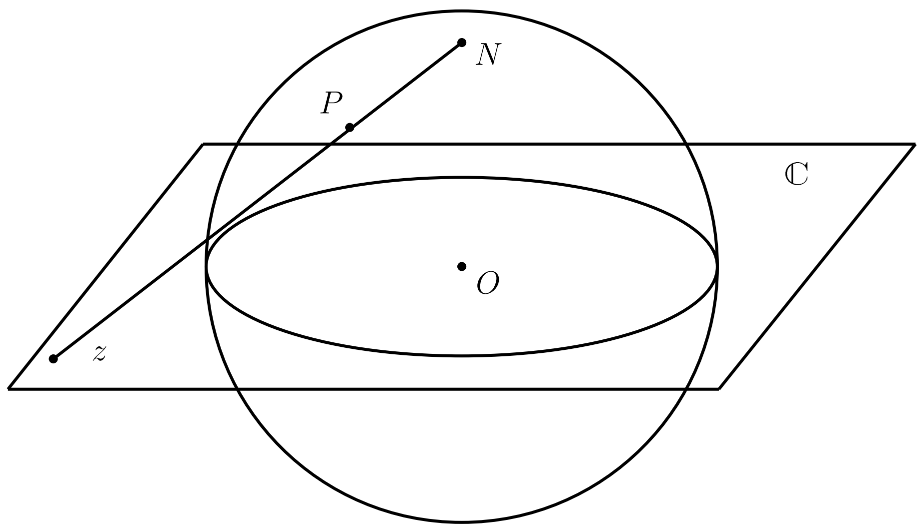
To find an inverse to $T(z) = Az + B$, we guess that $T^{-1}(z) = Cz + D$. Then we have $T^{-1} \circ T(z) = z$ which means that $C(Az+B)+D = z$. So we get $CAz+CB+D = z$, which means that $C = A^{-1}$, and $D = -AB^{-1}$. $\square$

Corollary 2.1. *A direct affine transformation preserves circles and lines.*

Proof. A direct affine transformation, $T(z) = Az+B$, where $|A| = 1$ is by Lemma 2.1 a rotation about $\frac{B}{1-A}$ which clearly preserves circles and lines. A direct affine transformation, $T(z) = Az + B$, where $|A| \neq 1$ can be written as $T(z) = r(A'z + B')$, where r is real and $|A'| = 1$. Again by Lemma 2.1 this map is a rotation about $\frac{B'}{1-A'}$ scaled by r, which preserves circles and lines. $\square$

Remark 2. A map that preserves angles is called conformal. By the same argument as above, a direct affine transformation is conformal.

3 The Stereographic Projection



Consider the unit sphere S^2 in $\mathbb{R}^3$, that is

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\}.$$

We embed S^2 such that the origin and the center of the sphere coincide. We identify the complex plane with the equitorial plane. That is, for coordinates (u, v, w) of $\mathbb{R}^3$, $\mathbb{C}$ is the plane where $w = 0$. The North pole has coordinates $N = (0, 0, 1)$. We will denote a complex number z by $x+iy$. Notice that the unit circle in $\mathbb{C}$ coincides with the equator of S^2.

For any point $P \in S^2$, there is a unique line from N to P, which we denote by NP. This line intersects the complex plane in exactly one point $z \in \mathbb{C}$.

Definition 2. The map from $SP \colon S^2 \setminus N \to \mathbb{C}$ which assigns to a point P the point $z \in \mathbb{C}$ given by the intersection $NP \cap \mathbb{C}$ is called the stereographic projection.

We will now derive the coordinate expressions for both stereographic projection and its inverse.

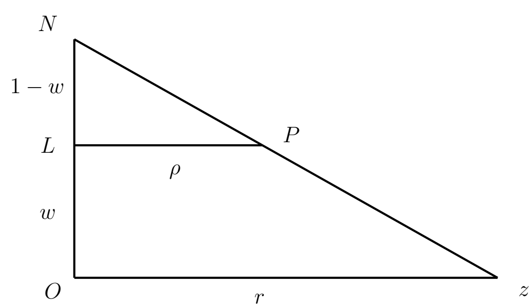
Proposition 3.1. *For $P \in S^2 \setminus N$ and $z \in \mathbb{C}$, $P = (u, v, w)$ and $z = x + iy$ the coordinates of the stereographic projection $SP\colon S^2 \setminus N \to \mathbb{C}$ and its inverse $SP^{-1}\colon \mathbb{C} \to S^2 \setminus N$ are given by:*

$$SP((u,v,w)) = \frac{u}{1-w} + i\frac{v}{1-w} \tag{1}$$

and

$$\begin{aligned} SP^{-1}(x+iy) &= \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right) &(2)\\ &= \left(\frac{z+\overline{z}}{z\overline{z}+1}, \frac{i(\overline{z}-z)}{z\overline{z}+1}, \frac{z\overline{z}-1}{z\overline{z}+1}\right). &(3)\end{aligned}$$

Proof.



From the picture we see that $|NO| = 1$, $|LO| = w$, $|LN| = 1 - w$, $|zO| = r = \sqrt{x^2+y^2}$. Set $|PL| = \rho$ and note that $\rho = \sqrt{u^2+v^2+w^2-w^2} = \sqrt{u^2+v^2}$.

Since the triangles NLP and NOZ are similar we have

$$\frac{\rho}{r} = \frac{1-w}{1} = \frac{u}{x} = \frac{v}{y}.$$

from which is follows that

$$\begin{aligned} \frac{1-w}{1} &= \frac{u}{x} \Rightarrow x = \frac{u}{1-w}\\ \frac{1-w}{1} &= \frac{v}{y} \Rightarrow y = \frac{v}{1-w}.\end{aligned}$$

This proves that $SP((u,v,w)) = \frac{u}{1-w} + i\frac{v}{1-w}$.

For the second statement we observe that since $z = x + iy$ we have

$$\begin{aligned} z &= \frac{u}{1-w} + i\frac{v}{1-w} = \frac{u+iv}{1-w}\\ \overline{z} &= \frac{u}{1-w} - i\frac{v}{1-w} = \frac{u-iv}{1-w}\end{aligned}$$

This implies that

$$\begin{aligned} x^2+y^2 &= z\overline{z} = \left(\frac{u+iv}{1-w}\right)\left(\frac{u-iv}{1-w}\right) = \\ &= \frac{u^2+v^2}{(1-w)^2} = \frac{(1-w)(1+w)}{(1-w)^2} = \\ &= \frac{1+w}{1-w} = -1+\frac{1}{1-w}. \end{aligned}$$

where we used $u^2+v^2 = 1-w^2$ in the fourth equality. This gives

$$1-w = \frac{2}{x^2+y^2+1} \Rightarrow w = \frac{x^2+y^2-1}{x^2+y^2+1} = \frac{z\overline{z}-1}{z\overline{z}+1}.$$

Since we have

$$x+iy = \frac{u+iv}{1-w} \Rightarrow 1-w = \frac{u+iv}{x+iy}$$

and we get

$$u+iv = (1-w)(x+iy) = \frac{2}{x^2+y^2+1}(x+iy) = \frac{2x+2iy}{x^2+y^2+1}.$$

Comparing real and imaginary parts gives

$$u = \frac{2x}{x^2+y^2+1} = \frac{z+\overline{z}}{z\overline{z}+1} v = \frac{2y}{x^2+y^2+1} = \frac{i(\overline{z}-z)}{z\overline{z}+1},$$

which finishes the proof. □

Definition 3. The extended complex plane is given by $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Remark 3. If we identify, via stereographic projection, points in the complex plane with points in $S^2 \setminus N$ and further identify ∞ with N then we have a bijection between the extended complex plane $\mathbb{C}_\infty$ and S^2. Under this identification S^2 is known as the Riemann sphere.

It is clear that stereographic projection is continuous as a map $S^2 \setminus N \to \mathbb{C}$ with a continuous inverse, since both maps are given as fractions of polynomials where the denominator is never zero. With the above identification of N with ∞ we get a continuous map $SP\colon S^2 \to \mathbb{C}_\infty$ with continuous inverse, $SP^{-1}\colon \mathbb{C}_\infty \to S^2$. A continuous map with continuous inverse is called a homeomorphism.

We note that since S^2 is compact and stereographic projection is a homeomorphism, $\mathbb{C}_\infty$ is compact as well. The space $\mathbb{C}_\infty$ is a[1] socalled one-point compactification of $\mathbb{C}$.

[1] There are other one-point compatifications of $\mathbb{C}$.

The stereographic projection gives a way of mapping a region of the sphere onto a plane. Using[2] the methods of Riemannian geometry, Gauss proved that it is impossible to find such a map that preserves both distances and angles. We will now see that the stereographic projection preserves angles, i.e. it is conformal.

Proposition 3.2. *The map* $SP\colon S^2 \to \mathbb{C}_\infty$ *given by stereographic projection and its inverse* $SP^{-1}\colon \mathbb{C}_\infty \to S^2$ *are conformal maps.*

Proof. Since the identity is conformal it is enough to prove that one of the maps is conformal, since the composition of two conformal maps is again conformal. We prove that $SP^{-1}\colon \mathbb{C}_\infty \to S^2$ preserves angles by showing that the angle between two lines in $\mathbb{C}$ is the same as the angle bewteen the two tangent vector at the point of intersection of the two curves lifted to S^2.

We can assume that one of the lines in $\mathbb{C}$ is the x-axis and that the intersect at an angle θ at the point p. That is consider the two lines $r(t) = p + t$ and $s(t) = t\,\mathrm{e}^{i\theta} + p$.

We have $SP^{-1}(z) = \left(\frac{z+\overline{z}}{z\overline{z}+1}, \frac{i(\overline{z}-z)}{z\overline{z}+1}, \frac{z\overline{z}-1}{z\overline{z}+1}\right)$, so we get

$$
\begin{aligned}
\overline{r}(t) &= SP^{-1}(r(t)) = \frac{(2(p+t), 0, (p+t)^2 - 1)}{1 + (p+t)^2} \\
\overline{s}(t) &= SP^{-1}(s(t)) = \\
&= \frac{(2(p + t\cos(\theta)), 2t\sin(\theta), (p + t\cos(\theta))^2 + t^2\sin^2(\theta) - 1)}{1 + (p + t\cos(\theta))^2 + t^2\sin^2(\theta)}.
\end{aligned}
$$

These two curves intersect at $t = 0$, so to find the angle between $\overline{r}(t)$ and $\overline{s}(t)$ we differentiate and evaluate at $t = 0$.

$$
\begin{aligned}
\overline{r}'(0) &= \frac{(2(p^2-1), 0, 4p)}{(p^2+1)^2} \\
\overline{s}'(0) &= \frac{(2(p^2-1)\cos(\theta), 2(p^2+1)\sin(\theta), 4p\cos(\theta))}{(p^2+1)^2}.
\end{aligned}
$$

We let φ be the angle between $\overline{r}'(0)$ and $\overline{s}'(0)$ and by calculating the dot product we

[2] A more accurate way of saying this is that by *inventing* Riemannian geometry Gauss proved a theorem usually referred to as Theorema Egregium, or, in English, totally awesome theorem, which states that it is impossible to construct a distance and angle preserving map between two spaces if the Gaussian curvature of the two space is not equal. In our case, $\mathbb{C}$ has curvature 0 and S^2 has curvature 1.

have

$$
\begin{aligned}
&\cos(\varphi) = \\
= \;& \frac{4(p^2-1)^2\cos(\theta)+16p^2\cos(\theta)}{\sqrt{(4(p^2-1)^2+16p^2)(4(p^2-1)^2\cos^2(\theta)+4(p^2+1)^2\sin^2(\theta)+16p^2\cos^2(\theta))}} \\
= \;& \frac{4(p^2+1)^2\cos(\theta)}{2(p^2+1)2(p^2+1)} = \\
= \;& \cos(\theta).
\end{aligned}
$$

So we see that $\theta = \varphi$, which means the angle is preserved. □

In the following it will be useful to have a clear definition of what is meant by a circle on S^2. A great circle on S^2 is given by the intersection of 2-dimensional subspace with S^2, and in general, a circle on S^2 is the intersection of a 2-dimensional plane with S^2. One extremely useful observation is the following.

Proposition 3.3. *Stereographic projection takes circles to circles and lines.*

Proof. Let us first consider the case where the circle goes through N. Let P be any point on the circle, then the line NP lies in the plane. Since the intersection of the plane with $\mathbb{C}$ is a line, stereographic projection takes the circle through N to a line in $\mathbb{C}$.

The other case is proved by considering the equation for the plane in $\mathbb{R}^3$, $Au + Bv + Cw + D = 0$, and express u, v, w in terms of x, y via stereographic projection and see that it defines a circle. From Proposition 3.1 we have $w = \frac{z\overline{z}-1}{z\overline{z}+1}$, $u = (1-w)x = \frac{2}{z\overline{z}+1}x$ and $v = (1-w)y = \frac{2}{z\overline{z}+1}y$. Plugging this into the equation for the plane we get

$$
\begin{aligned}
&A(1-w)x + B(1-w)y + Cw + D = 0 \\
&A\frac{2}{z\overline{z}+1}x + B\frac{2}{z\overline{z}+1}y + C\frac{z\overline{z}-1}{z\overline{z}+1} + D\frac{z\overline{z}+1}{z\overline{z}+1} = 0.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&2Ax + 2By + C(z\overline{z}-1) + D(z\overline{z}+1) = 0 \\
&2Ax + 2By + C(x^2+y^2-1) + D(x^2+y^2+1) = 0 \\
&(C+D)(x^2+y^2) + 2Ax + 2By + D - C = 0,
\end{aligned}
$$

which is a circle in $\mathbb{C}$ □

Remark 4. What does an open neighborhood around ∞ look like? A open neighborhood around $N \in S^2$ is the interior of a circle with center N in S^2. The circle is under stereographic projection mapped to a circle with the origin as its center of some radius, say r. This means that the interior of the circle around N is mapped to the points $\{z \in \mathbb{C}_\infty \mid |z| > r\}$. So since stereographic projection is a homeomorphism an open neighborhood around ∞ in $\mathbb{C}_\infty$ is the set $\{z \in \mathbb{C}_\infty \mid |z| > r\}$.

4 The Inversion Map

In this section we will study the map T, $T(z) = \frac{1}{z}$, called inversion. If $z \neq 0$ then there is a unique $w = \frac{1}{z}$, so $T\colon \mathbb{C} \setminus 0 \to \mathbb{C} \setminus 0$ is a bijection. Our goal is to extend T to a homeomorphism, $T\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$. Notice that $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$, so T is the composition of the two maps $S(z) = \overline{z}$ and $R(z) = \frac{z}{|z|^2}$. Furthermore we see that $\arg(R(z)) = \arg(z)$ and that $|R(z)| = \frac{1}{|z|}$, so R is inversion in the unit circle.

Proposition 4.1. *Let T denote the inversion. Under the identifications that $T(0) = \infty$ and $T(\infty) = 0$, T is a homeomorphism $T\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$.*

Proof. The map T is clearly a bijection $T\colon C_\infty \to \mathbb{C}_\infty$ so since T is continuous (it is the composition of the continuous maps R and S) as a map $T\colon C \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ all we need to check is that it is continuous at 0 and ∞. We have

$$\lim_{|z|\to 0} T(z) = \lim_{|z|\to\infty} T(\frac{1}{z}) = \lim_{|z|\to\infty} z = \infty,$$
$$\lim_{|z|\to\infty} T(z) = \lim_{|z|\to 0} T(\frac{1}{z}) = \lim_{|z|\to\infty} z = 0.$$

This shows that $T\colon C_\infty \to \mathbb{C}_\infty$ is continuous. Since the map T is its own inverse, it is a homeomorphism. $\square$

Proposition 4.2. *Let T denote the inversion. The map T takes circles and lines to circles and lines.*

Proof. We will not give all the details, since this is the same argument as in the proof of Proposition 3.3.

Let $w = \frac{1}{z}$ be the image of z under T. If $w = a + ib$ and $z = x + iy$ then we have $a = \frac{x}{x^2+y^2}$, $b = \frac{-y}{x^2+y^2}$, $x = \frac{a}{a^2+b^2}$ and $y = \frac{-b}{a^2+b^2}$. The equation

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

is a circle or, if $A = 0$, a line. Using the expressions for x, y in terms of a, b and substituting it into the equation for the circle or line, we get that a, b satisfy

$$D(a^2 + b^2) + Ba - Cb + A = 0$$

which is the equation for a circle or a line. $\square$

Remark 5. Note that if the circle goes through the origin it must be mapped to a line, since the image is unbounded, and a circle is bounded.

Example 1. Let us find the image of the vertical line $x = c_1$ under the inversion map. According to Equation 4, the image is the circle $-c_1(a^2 + b^2) + a = 0$ (put $A = 0$, $B = 1$, $C = 0$ and $D = -c_1$). This can be rewritten as $(a - \frac{1}{2}c_1) + b^2 = (\frac{1}{2}c_1)^2$, which a circle centered at $(\frac{1}{2}c_1, 0)$ of radius $\frac{1}{2}c_1$. Note that the circle goes through the origin, which we knew, since the image is a line.

Example 2. Let us find the image of the line $x = 0$ under the inversion map. Since $a = \frac{x}{x^2+y^2}$, we see that the image has x-coordinate equal to zero, hence the image lies on the imaginary axis. Now, $b = \frac{-y}{x^2+y^2} = \frac{-y}{y^2} = \frac{-1}{y}$. So the inversion takes $y \mapsto \frac{-1}{y}$ and maps $0 \mapsto \infty$.

Next we want to show that the inversion is a conformal map $\mathbb{C}_\infty \to \mathbb{C}_\infty$. This can be done in two different ways, we will show it geometrically by using the stereographic projection.

Remark 6. If we take $z = x + iy$ and $\overline{z} = x - iy$ and map them onto S^2 by SP^{-1}. How do the the points $SP^{-1}(z)$ and $SP^{-1}(\overline{z})$ differ? We use the equation for SP^{-1} to get

$$SP^{-1}(z) = SP^{-1}(x + iy) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right)$$
$$SP^{-1}(\overline{z}) = SP^{-1}(x - iy) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{-2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right)$$

This means that the two points $SP^{-1}(z)$ and $SP^{-1}(\overline{z})$ differ by a reflection in the u, w-plane. If we let $\Phi \colon S^2 \to S^2$ denote the reflection in the u, w-plane, we see that the composition $SP \circ \Phi \circ SP^{-1}$ corresponds to complex conjugation, i.e. $SP \circ \Phi \circ SP^{-1}(z) = \overline{z}$.

Similarly we can take the points z and $\frac{z}{|z|^2}$ (invert z in the unit circle) and map them onto S^2 by SP^{-1}, and ask how they differ. First note that since the unit circle is fixed under inversion in the unit circle, the corresponding map on S^2 must fix the equator. Furthermore, it must interchange the points N and S (the South pole), since the inversion in the unit circle maps 0 to ∞. A calculation shows that $SP^{-1}(z)$ and $SP^{-1}(\frac{z}{|z|^2})$ differ by a reflection in the equitorial plane. If we let $\Theta\colon S^2 \to S^2$ denote the reflection in the equitorial plane, we have that $SP \circ \Theta \circ SP^{-1}$ corresponds to inversion in the unit circle, i.e. $SP \circ \Theta \circ SP^{-1}(z) = \frac{z}{|z|^2}$. This means that as a map of S^2, the inversion map T is given by $\Phi \circ \Theta$ or as a map of $\mathbb{C}_\infty$ it can be written as

$$T = SP \circ \Phi \circ \Theta \circ SP^{-1}\colon \mathbb{C}_\infty \to \mathbb{C}_\infty.$$

Proposition 4.3. *The inversion $T\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a conformal map.*

Proof. Recall that the inversion is the composition of complex conjugation and inversion in the unit circle. Using the notation and results from the previous remark, the map $SP \circ \Phi \circ \Theta \circ SP^{-1}\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is equal to the inversion map, T. Since the maps SP, Φ, Θ and SP^{-1} are conformal (that reflections are conformal is easy to check), we see that T is conformal. □

5 Möbius Transformations

Having shown the basic properties of affine transformations, stereographic projection and the inversion map, we are now in a position to study Möbius transformations.

Definition 4. A Möbius transformation $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a map

$$f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

We have the following basic theorem about Möbius transformations.

Theorem 1. *Let f be any Möbius transformations, then*

1. *f can be expressed as a composition of affine transformations and inversions.*
2. *f maps $\mathbb{C}_\infty$ one-to-one onto itself, and is continuous.*
3. *f maps circles and lines to circles and lines.*
4. *f is conformal.*

Proof. 1. We write f as

$$\frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}.$$

If we let w_1, w_2 and w_3 be the maps $w_1 = cz + d$, $w_2 = \frac{1}{w_1}$ and $w_3 = (b - \frac{ad}{c})w_2 + \frac{a}{c}$, then $f = w_3 \circ w_2 \circ w_1$. Note that if $c = 0$, there is no inversion in the decomposition of f.

3. Since we have shown that both affine transformations and inversions take circles and lines to circles and lines, it follows from 1.) that f takes circles and lines to circles and lines.

2. If $z \neq -\frac{d}{c}$ and $w = \frac{az+b}{cz+d}$, then $z = \frac{-dw+b}{cw-a}$. So, at every point $z \in \mathbb{C}$, $z \neq -\frac{d}{c}$, f is well-defined, one-to-one, onto and continuous. We extend f to a map $f \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ by setting $f(\infty) = \infty$, if $c = 0$. If $c \neq 0$, then we set $f(-\frac{d}{c}) = \infty$. One can check that this makes f continuous as a function $f \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$. The inverse of f is given by

$$f^{-1}(w) = \frac{-dw + b}{cw - a}.$$

Again, if $c = 0$, we set $f^{-1}(\infty) = \infty$. If $c \neq 0$, then we set $f^{-1}(\frac{a}{c}) = \infty$. With these choices, one can check that $f^{-1} \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is continuous. In summary, $f \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a homeomorphism.

4. This also follows from 1. since we have shown that affine transformations and inversions are conformal, see Remark 2 and Proposition 4.3. □

Remark 7. As a map $f \colon \mathbb{C} \to \mathbb{C}$ it is continuous and conformal at every point $z \neq -\frac{d}{c}$. As a map $f \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ it is continuous and conformal everywhere.

One can determine if the image of a circle is a line or a circle by using the following argument. The map $f = \frac{az+b}{cz+d}$ is continuous as a map $\mathbb{C} \to \mathbb{C}$ at every point except $z = -\frac{d}{c}$ (this point is called a pole of f) and since a continuous function maps a bounded set to a bounded set, we conclude the if $z = -\frac{d}{c}$ does not lie on the circle, the image is bounded and hence a circle. If $z = -\frac{d}{c}$ lies on the circle, $z = -\frac{d}{c}$ is mapped to ∞ and the image is a line.

Definition 5. Let $\mathcal{M}(\mathbb{C}_\infty)$ denote the set of Möbius transformations

$$\mathcal{M}(\mathbb{C}_\infty) = \{f \colon \mathbb{C}_\infty \to \mathbb{C}_\infty \mid f(z) = \frac{az + b}{cz + d},\ ad - bc \neq 0\}.$$

This set has some additional structure; it is a group under composition of functions and can naturally be identified with $GL_2(\mathbb{C})$. We state this as a theorem.

Theorem 2. *The set $\mathcal{M}(\mathbb{C}_\infty)$ is a group under composition of functions and there is a surjective group homomorphism $\Upsilon \colon GL_2(\mathbb{C}) \to \mathcal{M}(\mathbb{C}_\infty)$ with kernel the diagonal matrices.*

Proof. We must show that the composition of two Möbius transformations is again a Möbius transformation. Let $f_1(z) = \frac{a_1z+b_1}{c_1z+d_1}$ and $f_2(z) = \frac{a_2z+b_2}{c_2z+d_2}$ be Möbius transformations. One easily checks that

$$f_2 \circ f_1(z) = \frac{(a_2a_1 + b_2c_1)z + a_2b_1 + b_2d_1}{(c_2a_1 + d_2c_1)z + c_2b_1 + d_2d_1},$$

which again a Möbius transformation.[3]

The identity $f(z) = z$ is a Möbius transformation. In the proof of Theorem 1 we saw that for a Möbius transformation $f(z) = \frac{az+b}{cz+d}$ the inverse is given by

$$f^{-1}(w) = \frac{-dw + b}{cw - a}.$$

Furthermore, the composition of functions is associative, so we have shown that the set $\mathcal{M}(\mathbb{C}_\infty)$ is a group under composition of functions.

We have a map $\Upsilon\colon GL_2(\mathbb{C}) \to \mathcal{M}(\mathbb{C}_\infty)$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{az+b}{cz+d}.$$

The map Υ sends the identity matrix to the map $f(z) = z$, which is the identity in the group $\mathcal{M}(\mathbb{C}_\infty)$. We must show that the product of two matrices is mapped to the product of two Möbius transformations. Let $f_1(z) = \frac{a_1z+b_1}{c_1z+d_1}$ and $f_2(z) = \frac{a_2z+b_2}{c_2z+d_2}$ be Möbius transformations.

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \end{bmatrix} \mapsto$$
$$\frac{(a_2a_1 + b_2c_1)z + a_2b_1 + b_2d_1}{(c_2a_1 + d_2c_1)z + c_2b_1 + d_2d_1} = f_2 \circ f_1(z).$$

This means that under the map $\Upsilon\colon GL_2(\mathbb{C}) \to \mathcal{M}(\mathbb{C}_\infty)$ matrix multiplication corresponds to composition of functions, which means that it is a homomorphism. It is onto since the Möbius transformation $f(z) = \frac{az+b}{cz+d}$ is hit by the matrix with the same entries. The kernel, $Ker(\Upsilon)$, is the subgroup of matrices that are mapped to the identity map. The Möbius transformation $f(z) = z$ can be written as $f(z) = \frac{az+0}{0z+a}$, which is exactly the image of the diagonal matrices. $\square$

[3] One should add that $f_2 \circ f_1$ cannot reduce to a constant. This is clear since the composition of two functions both of which are one-to-one and onto is again one-to-one and onto.

Remark 8. The kernel of Υ is $Ker(\Upsilon) = kI$, $k \in \mathbb{C}$ and I the identity matrix. Hence Υ induces a group isomorphism $\Upsilon\colon GL_2(\mathbb{C})/kI \to \mathcal{M}(\mathbb{C}_\infty)$. The group $GL_2(\mathbb{C})/kI$ is denoted $PGL_2(\mathbb{C})$.

If we let $SL_2(\mathbb{C})$ denote the complex matrices with determinant one, one can show that the map $\Upsilon\colon SL_2(\mathbb{C}) \to \mathcal{M}(\mathbb{C}_\infty)$ is onto, and has kernel $\pm I$, so one gets an isomorphism $\Upsilon\colon SL_2(\mathbb{C})/\pm I \to \mathcal{M}(\mathbb{C}_\infty)$. This just says that, instead of the condition $ad - bc \neq 0$, we could just as well have used $ad - bc = 1$.

We end this section with a few examples.

Example 3. Find the image of interior of the circle $|z-2| = 2$ under the map $f(z) = \frac{z}{2z-8}$. Note first that $z = 4$ is on the circle, so the image must be a line. We see that $f(0) = 0$ and $f(2+2i) = -\frac{i}{2}$, so the image is the imaginary axis. Since a continuous map preserves connectedness, the interior of the circle is either mapped to the right or left half plane. Since $f(2) = -\frac{1}{2}$, the set $\{z \in \mathbb{C} \mid |z-2| < 2\}$ is mapped to the set $\{z \in \mathbb{C} \mid \Re(z) < 0\}$.

Example 4. Construct a conformal map $\{z \in \mathbb{C} \mid |z| < 1\} \to \{z \in \mathbb{C} \mid \Re(z) > 0\}$. Since Möbius transformations are conformal, we construct a Möbius transformation that takes one set to the other. First we look for a map that takes the unit circle to the imaginary axis. The map must have a pole on the unit circle, since the imaginary axis is unbounded. Look at $f_1(z) = \frac{z+1}{z-1}$, which satisfies $f_1(1) = \infty$ and $f_1(-1) = 0$, so it maps the unit circle onto some straight line through the origin. Since $f_1(i) = -i$ it maps $\{z \in \mathbb{C} \mid |z| = 1\}$ to $\{z \in \mathbb{C} \mid \Re(z) = 0\}$. We see that the interior $\{z \in \mathbb{C} \mid |z| < 1\}$ is mapped to $\{z \in \mathbb{C} \mid \Re(z) < 0\}$, since $f_1(0) = -1$ and since f maps connected sets to connected sets. Hence the map $f(z) = -\frac{z+1}{z-1}$ maps $\{z \in \mathbb{C} \mid |z| < 1\} \to \{z \in \mathbb{C} \mid \Re(z) > 0\}$.

Example 5. Determine the image of the second quadrant $\{z \in \mathbb{C} \mid \Re(z) < 0 \text{ and } \Im(z) > 0\}$ under the mapping $f(z) = \frac{z+i}{z-i}$. Let us first see where the two axis are mapped. Since $f(2i) = 3$ and $f(3i) = 2$ the imaginary axis is mapped to the real axis. We have $f(0) = -1$ and $f(-1) = -i$, so the imaginary axis is mapped to the line $\{z = x+iy \mid y = -x-1\}$. We also have $f(-1+3i) = \frac{9-2i}{5}$, so we see that f maps the set $\{z \in \mathbb{C} \mid \Re(z) < 0 \text{ and } \Im(z) > 0\}$ to the set $\{z = x+iy \mid y > -x-1 \text{ and } y < 0\}$.

6 The Cross Ratio

We have already seen that Möbius transformations map circles to circles[4]. In this section we want to find a specific Möbius transformation that takes a specific circle

[4]In this section we will not destinguish between circles and lines, since a line in $\mathbb{C}$ is a circle in $\mathbb{C}_\infty$ closed at ∞.

to another specific circle. Recall from Euclidean geometry that three points uniquely determine a circle. Let us denote one circle by C_1 and one by C_2. We choose points z_1, z_2 and z_3 on C_1 and w_1, w_2 and w_3 on C_2. Then if we find a Möbius transformation h that takes

$$h(z_1) = w_1, \quad h(z_2) = w_2, \quad h(z_3) = w_3 \tag{4}$$

then h must map C_1 to C_2. The trick is first to map C_1 onto the real axis, then map the real axis onto C_2. To map C_1 onto the real axis is the same as solving Equation 4 for $w_1 = 0$, $w_2 = 1$ and $w_3 = \infty$.

If the points $z_i \neq \infty$, we define a Möbius transformation f by

$$f(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \tag{5}$$

which clearly takes $f(z_1) = 0$, $f(z_2) = 1$, $f(z_3) = \infty$. If one of the three points $z_i = \infty$ (which means that C_1 is a line) we have

$$f(z) = \frac{z_2 - z_3}{z - z_3}\ (z_1 = \infty), \quad f(z) = \frac{z - z_1}{z - z_3}\ (z_2 = \infty), \quad f(z) = \frac{z - z_1}{z_2 - z_1}\ (z_1 = \infty) \tag{6}$$

which satisfy $f(z_1) = 0$, $f(z_2) = 1$, $f(z_3) = \infty$. Now let g be another Möbius transformation which takes $g(w_1) = 0$, $g(w_2) = 1$, $g(w_3) = \infty$. Then we see that the map $h = g^{-1} \circ f$ satisfies

$$\begin{aligned} h(z_1) &= g^{-1} \circ f(z_1) = g^{-1}(0) = w_1 \\ h(z_2) &= g^{-1} \circ f(z_2) = g^{-1}(1) = w_2 \\ h(z_3) &= g^{-1} \circ f(z_3) = g^{-1}(\infty) = w_3. \end{aligned}$$

Notice that the equation $h(z) = w$ can be written as

$$g^{-1}(f(z)) = w \quad \Longleftrightarrow \quad g(w) = f(z) \tag{7}$$

which means that

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}. \tag{8}$$

These fractions are called cross ratios.

Definition 6. Let z, z_1, z_2, and z_3 be four points in $\mathbb{C}_\infty$. Then the expression

$$(z, z_1, z_2, z_3) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

is called the cross ratio of the four points z, z_1, z_2, and z_3.

To map z_1, z_2, and z_3 onto w_1, w_2, and w_3 we have to solve Equation 7 for w in terms of z, which by Equation 8 is the same as equating the two cross ratios and solving for w in terms of z. Note that the order of the terms in the cross ratio is essential.

We now give some examples.

Example 6. Find a Möbius transformation that takes 0 to i, 1 to 2 and -1 to 4.

We calculate the approprate cross ratios

$$\begin{aligned}(z, 0, 1, -1) &= \frac{(z-0)(-1-(-1)}{(z-(-1))(1-0)} = \frac{2z}{z+1}\\ (w, i, 2, 4) &= \frac{w-i(2-4)}{(w-4)(2-i)} = \frac{-2(w-i)}{(w-4)(2-i)}.\end{aligned}$$

So to find the Möbius transformation we must equate the two expressions and solve for w.

$$\frac{-2(w-i)}{(w-4)(2-i)} = \frac{2z}{z+1}$$

which gives

$$w = h(z) = \frac{(16-6i)z+2i}{(6-2i)z+2}$$

which is the desired Möbius transformation.

The next example is analogous to an earlier example which we treated by an ad hoc method. The concept of an orientation gives us an algorithm to solve similar problems.

Remark 9. A circle is not only determined by the three points z_1, z_2 and z_3 but also oriented by the three points. A line also needs three points to orient it, since it closes up at ∞. The orientation is given by proceeding through the three points in succession. This orientation determines a left region, namely the region that is to the left of an observer traversing the circle in the direction of the orientation. Using conformality of the Möbius transformation it can be shown that given two circles C_1 and C_2 with orientations determined by the points z_1, z_2 and z_3 and w_1, w_2 and w_3, the Möbius transformation maps the left region of C_1 to the left region of C_2.

Example 7. Find a Möbius transformation that takes the region $D_1 = \{z \in \mathbb{C} \mid |z| > 1\}$ to the region $D_2 = \{z \in \mathbb{C} \mid \Re(z) < 0\}$.

We choose both D_1 and D_2 to be left regions. That is accomplished by choosing $z_1 = 1$, $z_2 = -i$, $z_3 = -1$ and $w_1 = 0$, $w_2 = i$, $w_3 = \infty$. Since Möbius transformations take left regions to left regions, a solution to the problems is a Möbius transformation that takes 1 to 0, $-i$ to i and -1 to ∞. As in the previous example we find such a Möbius transformation by equating the two cross ratios, i.e.

$$(w, 0, i, \infty) = (z, 1, -i, -1)$$

which is the same as

$$\frac{w - 0}{i - 0} = \frac{(z - 1)(-i + 1)}{(z + 1)(-i - 1)},$$

where we have used the first formula in Equation 6 to calculate the cross ratio for w. This gives the desired Möbius transformation

$$w = h(z) = \frac{1 - z}{1 + z}.$$

7 The Symmetry Principle and Maps of the Unit Disk and the Upper Halfplane

We wish to determine all possible Möbius transformations from the unit disk to itself, and from the upper halfplane to itself. For this, we need the symmetry principle. First we need a few definitions.

Definition 7. Two points z_1 and z_2 are symmetric with respect to a straight line L if L is the perpendicular bisector of the line joining z_1 and z_2.

A circle is orthogonal to a line L if the tangent of the circle is orthogonal to the line at the point of intersection. This is equivalent to the center of the circle lying on L.

Definition 8. Two points z_1 and z_2 are symmetric with respect to a circle C if every straight line or circle passing through z_1 and z_2 intersects C orthogonally.

Note that this in particular means that the center a of C and ∞ are symmetric with respect to C. Since a circle is bounded there is no circle through both a and ∞ and any line through a (and ∞) is orthogonal to C. We can now state the symmetry-preserving properties of Möbius transformations.

Theorem 3 (The Symmetry Principle)**.** *Let C be a line or circle in $\mathbb{C}$ and let f be a Möbius transformation. Two points z_1 and z_2 are symmetric with respect to C if and only if their images under f, $f(z_1)$ and $f(z_2)$ are symmetric with respect to the image of C under f, $f(C)$.*

Proof. Since Möbius transformations are conformal they preserve orthogonality. Two points are symmetric with respect to a circle or line if every circle or line containing the points intersects the given circle or line orthogonally. Since they preserve the class of circles and lines and preserve orthogonality, they preserve the symmetry condition. $\square$

Given a circle C and a point α we would like a formula for the point α^*, the point symmetric to α with respect to C.

Proposition 7.1. *Given a point α and a circle C with center a and radius R. Then*

$$\alpha^* = \frac{R^2}{\overline{\alpha} - \overline{a}} + a, \tag{9}$$

is the point symmetric to α with respect to C.

Proof. By equating the cross ratios $(w, 0, 1, \infty)$ and $(z, a - R, a + Ri, a + R)$ we observe that points $a - R$, $a + Ri$ and $a + R$ (and hence all of C) is mapped to the real axis by

$$f(z) = i\frac{z - (a - R)}{z - (a + R)}. \tag{10}$$

By the symmetry principle α is symmetric to α^* with respect to C if and only if $f(\alpha)$ is symmetric to $f(\alpha^*)$ with respect to the real axis. That $f(\alpha)$ is symmetric to $f(\alpha^*)$ with respect to the real axis is clearly equivalent to $f(\alpha)$ and $f(\alpha^*)$ are complex conjugate points in $\mathbb{C}$, $f(\alpha) = \overline{f(\alpha^*)}$. Using Equation 10 this is equivalent to

$$i\frac{\alpha^* - (a - R)}{\alpha^* - (a + R)} = \overline{i\frac{z - (a - R)}{z - (a + R)}} = -i\frac{\overline{\alpha} - (\overline{a} - R)}{\overline{\alpha} - (\overline{a} + R)}$$

which when solving for α^* yields

$$\alpha^* = \frac{R^2}{\overline{\alpha} - \overline{a}} + a.$$

This also shows that the point α^* is unique. $\square$

Remark 10. From Equation 9 we see that

$$\arg(\alpha^* - a) = \arg\left(\frac{R^2}{\overline{\alpha} - \overline{a}}\right) = \arg\left(\frac{R^2(\alpha - a)}{|\alpha - a|^2}\right) = \arg(\alpha - a).$$

This means that the symmetric points lie on the same line from the center a. Furthermore we have

$$|\alpha^* - a| = \frac{R^2}{|\alpha - a|} \Rightarrow R^2 = |\alpha^* - a||\alpha - a|.$$

Next we classify the Möbius transformations that take the unit disk to itself[5]. We denote the open unit disk by D^2.

Theorem 4. *Let f be a Möbius transformation that takes D^2 to itself, then*

$$f(z) = \mathrm{e}^{i\theta}\,\frac{z - \alpha}{\overline{\alpha} z - 1}, \quad \textit{where } \alpha \in \mathbb{C} \textit{ and } |\alpha| < 1.$$

Proof. Let f be such a Möbius transformation. Then f maps the unit circle C_z to the unit circle C_w. Furthermore, since the interior is mapped to the interior there must be a point α, $|\alpha| < 1$ which is mapped to the origin, i.e. $f(\alpha) = 0$. According to Equation 9 with $a = 0$ the point

$$\alpha^* = \frac{1^2}{\overline{\alpha} - \overline{0}} = \frac{1}{\overline{\alpha}}$$

is symmetric to α with respect to C_z. By the symmetry principle this implies that $f(\frac{1}{\overline{\alpha}})$ is symmetric to $f(\alpha) = 0$ with respect to C_w. Since the origin is the center of C_w its symmetric point is ∞. This implies

$$f\left(\frac{1}{\overline{\alpha}}\right) = \infty.$$

This means that f has a zero at α and a pole at $\frac{1}{\overline{\alpha}}$. This implies that f is of the form

$$f(z) = k\frac{z - \alpha}{z - \frac{1}{\overline{\alpha}}} = k\overline{\alpha}\frac{z - \alpha}{\overline{\alpha} z - 1}$$

for some constant k.

[5]More generally it can be shown that the Möbius transformations are the one-to-one analytic (complex differentiable) maps of the unit disk to itself.

We also know that $f(1)$ must be mapped to some point on C_w, so we have

$$1 = |f(1)| = |k\overline{\alpha}|\frac{|1-\alpha|}{|\overline{\alpha}-1|} = |k\overline{\alpha}|.$$

This implies that $k\overline{\alpha} = \mathrm{e}^{i\theta}$ for some $\theta \in [0, 2\pi]$. We see that f must look like

$$f(z) = \mathrm{e}^{i\theta}\,\frac{z-\alpha}{\overline{\alpha}z-1}, \quad \text{where } |\alpha| < 1.$$

Now we just need to check that all such maps takes the disk to itself. Let $\alpha = a + ib$, then we have

$$\begin{aligned}
|f(i)| &= \left|\mathrm{e}^{i\theta}\,\frac{i-\alpha}{\overline{\alpha}i-1}\right| = \frac{|i-\alpha|}{|\overline{\alpha}i-1|} = \frac{|-a+(1-b)i|}{|b-1+ai|} = \frac{\sqrt{(-a)^2+(1-b)^2}}{\sqrt{(b-1)^2+a^2}} = 1,\\
|f(1)| &= \left|\mathrm{e}^{i\theta}\,\frac{1-\alpha}{\overline{\alpha}-1}\right| = 1,\\
|f(-1)| &= \left|\mathrm{e}^{i\theta}\,\frac{-1-\alpha}{-\overline{\alpha}-1}\right| = 1.
\end{aligned}$$

Since $f(\alpha) = 0$ and $|\alpha| < 1$ the interior goes to the interior, which proves that f maps the disk to itself. $\square$

We finish this section with a classification of the Möbius transformations that takes $\mathbb{H}^2$, $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, to $\mathbb{H}^2$. We formulate this in the following theorem.

Theorem 5. *The Möbius transformations $\mathbb{H}^2 \to \mathbb{H}^2$ are the maps*

$$f(z) = \frac{az+b}{cz+d}, \ a, b, c, d \in \mathbb{R} \textit{ and } ad - bc > 0.$$

For the proof we need the following proposition.

Proposition 7.2. *A Möbius transformation f maps the real line to the real line if and only if the coefficients a, b, c and d are real.*

Proof. We first assume that the coefficients a, b, c and d of f are real. Then the image of the three points 0, 1 and 2 is clearly three points on the real axis. Since the image of a line is determined by the image of three points on that line, this proves that the real axis is mapped to the real axis.

Denote the real axis by $\mathbb{R}_\infty$[6]. Assume now that $f(\mathbb{R}_\infty) = \mathbb{R}_\infty$. This implies that f maps three points on the real axis q_1, q_2 and q_3 to three points on the real axis r_1, r_2 and r_3 (we assume that all points are finite, if this is not the case, modify the following cross ratios acordingly, see Equation 6). This means that f is determined by equating the two cross ratios

$$(w, r_1, r_2, r_3) = (z, q_1, q_2, q_3)$$

which is the same as

$$\frac{(z-q_1)(q_2-q_3)}{(z-q_3)(q_2-q_1)} = \frac{(w-r_1)(r_2-r_3)}{(w-r_3)(r_2-r_1)}.$$

When we solve for w we get a Möbius transformation $w = f(z)$ with real coefficients, since q_1, q_2, q_3, r_1, r_2 and r_3 are real. Since the points q_1, q_2, q_3, r_1, r_2 and r_3 were arbitrary, this finishes the proof. □

This proposition allows us to give a quick proof of the theorem.

Proof. Let f be a Möbius transformation that maps $\mathbb{H}^2$ to $\mathbb{H}^2$. Since the real axis partitions $\mathbb{C}_\infty$ into two connected components, and a Möbius transformation maps a connected component to a connected component, f must map the real axis to the real axis. By Proposition 7.2 we can choose f as

$$f(z) = \frac{az+b}{cz+d}, \ a, b, c, d \in \mathbb{R}$$

We have

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \\ &= \frac{az+b}{|cz+d|^2}(c\overline{z}+d) \\ &= \frac{1}{|cz+d|^2}(ac|z|^2 + bd + bc\overline{z} + adz). \end{aligned}$$

Hence we have

$$\Im(f(z)) = \Im\left(\frac{az+b}{cz+d}\right) = \Im\left(\frac{1}{|cz+d|^2}(ac|z|^2 + bd + bc\overline{z} + adz)\right) = \frac{ad-bc}{|cz+d|^2}\Im(z).$$

This means that f maps $\mathbb{H}^2$ to itself if and only if $ad - bc > 0$. □

[6]The subscript is a reminder that this is a circle in $\mathbb{C}_\infty$ which is closed up at ∞

Remark 11. The set of maps

$$\mathcal{M}_{\mathbb{R}}(\mathbb{C}_\infty) = \left\{ f(z) = \frac{az+b}{cz+d} \mid \ a,b,c,d \in \mathbb{R} \right\}$$

is a subgroup of the the group of Möbius transformations which analogously to the case where the coefficients are complex can be identified with the set of matrices $GL_2(\mathbb{R})/kI = PGL_2(\mathbb{R})$. The subgroup of matrices in $GL_2(\mathbb{R})$ with positive determinant is denoted by $GL_2^+(\mathbb{R})$. We have a surjective map

$$\Upsilon \colon GL_2^+(\mathbb{R}) \to \left\{ \frac{az+b}{cz+d} \mid \ a,b,c,d \in \mathbb{R} \text{ and } ad-bc>0 \right\} = \mathcal{M}_{\mathbb{R}}^+(\mathbb{C}_\infty)$$

with $Ker(\Upsilon) = kI$, where $k \in \mathbb{R}$, $k > 0$, and I is the identity matrix. Hence we get an isomorphism

$$\Upsilon \colon GL_2^+(\mathbb{R})/kI \to \mathcal{M}_{\mathbb{R}}^+(\mathbb{C}_\infty)$$

The group $GL_2^+(\mathbb{R})/kI$ is denoted $PGL_2^+(\mathbb{R})$.

One can ask, if there are more conformal bijections of D^2 and $\mathbb{H}^2$ than the ones we have determined in the preceeding theorems. The answer relies on some difficult results in complex analysis, so we state the theorem without a proof.

Theorem 6. *The conformal maps of $\mathbb{C}_\infty$ are precisely the Möbius transformations. In particular, the maps $D^2 \to D^2$ and $\mathbb{H}^2 \to \mathbb{H}^2$ determined in Theorem 4 and Theorem 5 are the only conformal bijections of D^2 and $\mathbb{H}^2$.*

Since $\mathbb{H}^2$ and D^2 both serve as models of hyperbolic geometry, it is important to find all conformal maps from $\mathbb{H}^2 \to D^2$. The map [7]

$$\varphi(z) = \frac{z-i}{z+i} \tag{11}$$

is a conformal bijection $\mathbb{H}^2 \to D^2$ and we use this map and the calculation we have done for $\mathbb{H}^2$ to determine all conformal bijections $\mathbb{H}^2 \to D^2$. We state the result as a theorem.

Theorem 7. *The set of conformal bijections $g \colon \mathbb{H}^2 \to D^2$ is the set of maps*

$$\{g = \varphi \circ h \mid h(z) = \frac{az+b}{cz+d}, \ a,b,c,d \in \mathbb{R} \text{ and } ad-bc>0\} \tag{12}$$

where φ is given by Equation 11. This set is in one-to-one correspondence with the group $PGL_2^+(\mathbb{R})$

[7] The are of course many other conformal bijections $\mathbb{H}^2 \to D^2$.

Proof. Since φ is a conformal bijection $\mathbb{H}^2 \to D^2$ it is in view of Theorem 6 clear that all possible conformal bijections $\mathbb{H}^2 \to D^2$ are given by Equation 12. A bijection

$$\{g = \varphi \circ h \mid h(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{R} \text{ and } ad-bc > 0\} \to$$
$$\{h \mid h(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{R} \text{ and } ad-bc > 0\}$$

is given by $\varphi \circ h \mapsto h$, with inverse $h \mapsto \varphi \circ h$. We have previously noted that the set of conformal bijections $\mathbb{H}^2 \to \mathbb{H}^2$ is $PGL_2^+(\mathbb{R})$. □

Remark 12. Since $\varphi\colon \mathbb{H}^2 \to D^2$ is a conformal bijection, the set of conformal bijections of D^2 is in bijective correspondence with the set of conformal bijections of $\mathbb{H}^2$. If we let f be any conformal bijection of $\mathbb{H}^2$ and g be any conformal bijection of D^2, then the bijection is given by conjugating with φ and φ^{-1}, that is, $f \mapsto \varphi \circ f \circ \varphi^{-1}$ and $g \mapsto \varphi^{-1} \circ g \circ \varphi$. For this reason the set of conformal bijections of D^2 must be in one-to-one correspondence with $PGL_2^+(\mathbb{R})$, which is not obvious at all from the description in Theorem 4.

8 Conjugacy Classes in $\mathcal{M}(\mathbb{C}_\infty)$

In this section we will work in the group $PSL_2(\mathbb{C})$, which is isomorphic to $\mathcal{M}(\mathbb{C}_\infty)$. The assumption "Let $f \in PSL_2(\mathbb{C})$" means that f is a Möbius transformation which corresponds to an equivalence class in $PSL_2(\mathbb{C})$ given by $\pm A$, $A \in SL_2(\mathbb{C})$.

Definition 9. Two elements $f, g \in PSL_2(\mathbb{C})$ are conjugate if there exists $h \in PSL_2(\mathbb{C})$ such that $f = h \circ g \circ h^{-1}$.

Remark 13. Note that conjugacy is an equivalence relation, which partitions $PSL_2(\mathbb{C})$ into disjoint equivalence classes. The equivalence class containing the identity map, id, contains no other elements, since for all $g \in PSL_2(\mathbb{C})$ we have $g \circ id \circ g^{-1} = g \circ g^{-1} = id$.

The following easy fact will be used again and again in the sequel: If z_0 is a fixed point of f then $h(z_0)$ is a fixed point of $h \circ f \circ h^{-1}$, and the maps f and $h \circ f \circ h^{-1}$ have the same number of fixed points.

We start with the following theorem.

Theorem 8. *Let $f \in PSL_2(\mathbb{C})$, $f(z) = \frac{az+b}{cz+d}$. If $(a+d)^2 \neq 4$, then f has two fixed points in $\mathbb{C}_\infty$. If $(a+d)^2 = 4$, f not equal to the identity, then f has one fixed point in $\mathbb{C}_\infty$.*

Proof. The map f fixes ∞ if and only if $c = 0$. If $c \neq 0$, z is a fixed point if and only if it is a root in $cz^2 + (d-a)z - b = 0$. This has two roots unless the discriminant $(d-a)^2 + 4bc = 0$. Now we have

$$\begin{aligned}(d-a)^2 + 4bc = 0 \qquad &\iff a^2 + d^2 - 2ad + 4bc = 0\\ &\iff a^2 + d^2 - 2ad + 4bc + 4 = 4\\ &\iff a^2 + d^2 - 2ad + 4bc + 4(ad - bc) = 4\\ &\iff a^2 + d^2 + 2ad = 4.\end{aligned}$$

This implies that if $c \neq 0$, then f has a single fixed point if and only if $(a+d)^2 = 4$.

If $c = 0$, then $ad = 1$ and we know that $f(z) = \frac{az+b}{d} = a^2 z + ba$, since $ad = 1$. This map has ∞ as a fixed point and the second fixed point is $\frac{ab}{1-a^2}$. The point $\frac{ab}{1-a^2} \neq \infty$ if and only if and only if $a^2 \neq 1$ or equivalently $(a+d)^2 \neq 4$, since $ad = 1$. If $a^2 = 1$ then $f(z) = z \pm b$, so either $f(z) = z$ or f has ∞ as its only fixed point. $\square$

For a matrix $A \in M_{2\times 2}(\mathbb{C})$ we have the trace

$$tr\colon M_{2\times 2}(\mathbb{C}) \to \mathbb{C} \quad \text{given by} \quad tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

For two matrices A and B we have

$$\begin{aligned}&tr(AB) = tr(BA)\\ &tr(ABA^{-1}) = tr(B)\\ &tr(-A) = -tr(A).\end{aligned}$$

In particular we see that the trace only depends on the conjugacy class of a matrix. Since an equivalence class in $PSL_2(\mathbb{C})$ is represented by the matrices $\pm A$, $A \in SL_2(\mathbb{C})$, we see that the trace is not well-defined as a function $tr\colon PSL_2(\mathbb{C}) \to \mathbb{C}$. However, when we square the trace we get a well-defined function

$$tr^2\colon PSL_2(\mathbb{C}) \to \mathbb{C}.$$

We state this as a propositon.

Proposition 8.1. *Let $f \in PSL_2(\mathbb{C})$, $f = \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The function $tr^2\colon PSL_2(\mathbb{C}) \to \mathbb{C}$, given by*

$$tr^2\left(\pm \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d)^2.$$

is well-defined and depends only on the conjugacy class of f.

Example 8. Let $f_\lambda(z) = \lambda z$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Then f_λ is represented in $SL_2(\mathbb{C})$ by the matrices

$$f_\lambda = \pm \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix}$$

and so we have $tr^2(f_\lambda) = (\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}})^2 = \lambda + \frac{1}{\lambda} + 2$.

Definition 10. Define the functions $f_\lambda \in PSL_2(\mathbb{C})$ by

$$f_\lambda(z) = \begin{cases} \lambda z & \lambda \neq 0, 1, \\ z + 1 & \lambda = 1. \end{cases}$$

These simple maps turn out to represent all the conjugacy classes in $PSL_2(\mathbb{C})$. We state this as a theorem.

Theorem 9. *If f is a non-identity element in $PSL_2(\mathbb{C})$, then there exists some $\lambda \in \mathbb{C} \setminus \{0\}$ such that f is conjugate to f_λ.*

Proof. Suppose first that f has only one fixed point, z_0. Using cross ratios we find a g such that $g(z_0) = \infty$. The map $g \circ f \circ g^{-1}$ fixes only $g(z_0) = \infty$, since f only has one fixed point. Since ∞ is the only fixed point we must have $g \circ f \circ g^{-1}(z) = z + t$ for some t. If we let $h(z) = \frac{z}{t}$ we see that

$$h \circ g \circ f \circ g^{-1} \circ h^{-1}(z) = z + 1$$

which means that $(h \circ g) \circ f \circ (h \circ g)^{-1}(z) = z + 1$, and so f is conjugate to f_1.

If f has two fixed points z_1 and z_2, we use cross ratios to find a map g such that $g(z_1) = 0$ and $g(z_2) = \infty$. This implies that $g \circ f \circ g^{-1}$ fixes 0 and ∞ and hence is a dilation, that is, $g \circ f \circ g^{-1}(z) = \lambda z$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$. This means that f is conjugate to f_λ, $\lambda \in \mathbb{C} \setminus \{0, 1\}$. □

To determine the conjugacy classes completely we must see when two maps f_λ and f_κ are conjugate.

Theorem 10. *The maps f_λ and f_κ are conjugate if and only if $\lambda = \kappa$ or $\lambda = \frac{1}{\kappa}$, i.e. $tr^2(f_\lambda) = tr^2(f_\kappa)$.*

Proof. First we consider the case $\lambda = 1$. Since f_1 fixes only ∞, the map $g \circ f_1 \circ g^{-1}$ fixes only $g(\infty)$. This implies that f_1 is not conjugate to f_κ, $\kappa \neq 1$, since f_κ, $\kappa \neq 1$ fixes both 0 and ∞.

Now we let $\lambda \neq 1$ and $\kappa \neq 1$. Suppose that f_λ and f_κ are conjugate, then $tr^2(f_\lambda) = tr^2(f_\kappa)$. This implies $\lambda + \frac{1}{\lambda} + 2 = \kappa + \frac{1}{\kappa} + 2$, which implies that $\lambda = \kappa$ or $\lambda = \frac{1}{\kappa}$.

On the other hand, if $T(z) = \frac{1}{z}$ we have

$$T \circ f_\lambda \circ T^{-1}(z) = \frac{1}{\lambda \frac{1}{z}} = \frac{1}{\lambda} z = f_{\frac{1}{\lambda}}(z)$$

which means that f_λ is conjugate to $f_{\frac{1}{\lambda}}$. □

Corollary 8.1. *Two non-identity elements $f, g \in PSL_2(\mathbb{C})$ are conjugate if and only if $tr^2(f) = tr^2(g)$.*

Proof. We already know that if f and g are conjugate, then $tr^2(f) = tr^2(g)$.

Now assume $tr^2(f) = tr^2(g)$ and pick representatives f_λ and f_κ in the conjugacy classes of f and g. This means that there exists $h_1, h_2 \in PSL_2(\mathbb{C})$ such that $f_\lambda = h_1 \circ f \circ h_1^{-1}$ and $f_\kappa = h_2 \circ g \circ h_2^{-1}$. Since $tr^2(f) = tr^2(g)$ we have $tr^2(f_\lambda) = tr^2(f_\kappa)$, and so by the previous theorem f_λ and f_κ are conjugate. Now f is conjugate to f_λ, f_λ is conjugate to f_κ and f_κ is conjugate to g, which by transitivity of the relation implies that f is conjugate to g. □

Definition 11. Let $f \in PSL_2(\mathbb{C})$. Then f is conjugate to f_λ and $f_{\frac{1}{\lambda}}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. The pair $(\lambda, \frac{1}{\lambda})$ is called the multiplier of f.

Remark 14. It follows from the theorem that two maps are conjugate if and only if they have the same multiplier. The multiplier and the tr^2 are related by the fact that $(\lambda, \frac{1}{\lambda})$ are roots in the equation $z^2 + (2 - tr^2(f))z + 1 = 0$.

9 Geometric Classification of Conjugacy Classes

In this section we will study the limiting behaviour of representatives in the different conjugacy classes. The behaviour is determined by the multiplier, $(\lambda, \frac{1}{\lambda})$ or equivalently by $tr^2(f) = \lambda + \frac{1}{\lambda} + 2$, since f is conjugate to f_λ if and only if $tr^2(f) = tr^2(f_\lambda) = \lambda + \frac{1}{\lambda} + 2$.

Let $f \in PSL_2(\mathbb{C})$ and assume first that $tr^2(f) = 4$. This is equivalent to the multiplier of f being $(1, 1)$ or $\lambda = 1$ and hence f is conjugate to f_1. Such an f has only one fixed point, z_0. We let $g \in PSL_2(\mathbb{C})$ be a map such that $g(z_0) = \infty$. Then the conjugate $h = g \circ f \circ g^{-1}$ fixes only ∞. That means h is a translation and since there is only one fixed point, we have $h(z) = z + k$, $k \neq 0$. It follows that $h^n(z) = z + nk$. We have

$$\lim_{n\to\infty} h^n(z) = \lim_{n\to\infty} z + nk = \infty$$

for all $z \in \mathbb{C}$. This implies

$$\lim_{n\to\infty} h^n(z) = \lim_{n\to\infty} g \circ f^n \circ g^{-1}(z)$$

which implies

$$\lim_{n\to\infty} g^{-1} \circ h^n(z) = \lim_{n\to\infty} g^{-1} \circ g \circ f^n \circ g^{-1}(z).$$

By composing with g we get

$$\lim_{n\to\infty} g^{-1} \circ h^n \circ g(z) = \lim_{n\to\infty} f^n \circ g^{-1} \circ g(z) = \lim_{n\to\infty} f^n(z).$$

Since $\lim_{n\to\infty} h^n(z) = \lim_{n\to\infty} z + nk = \infty$ for all $z \in \mathbb{C}$ we have $\lim_{n\to\infty} h^n \circ g(z) = \infty$. This implies

$$\lim_{n\to\infty} f^n(z) = g^{-1}(\infty) = z_0$$

for all $z \in \mathbb{C}$. Hence we see that by applying f repeatedly, all points in $\mathbb{C}$ are moved towards the fixed point, z_0. A map with $\lambda = 1$ is called *parabolic.*

Next we consider the case where $tr^2(f) \neq 4$. In this case f is conjugate to f_λ, $\lambda \neq 0, 1$, and f has two fixed points, z_1 and z_2.

If $|\lambda| = 1$ then $\lambda = \mathrm{e}^{i\theta}$. This means that the limit of f_λ^n does not exist and hence neither does the limit of f^n. Such a map is called *elliptic.*

There are two case remaining: $|\lambda| < 1$ and $|\lambda| > 1$.

Using cross ratios we find a map g that takes z_1 to 0 and z_2 to ∞. Then $g \circ f \circ g^{-1}$ fixes 0 and ∞ and hence $g \circ f \circ g^{-1} = f_\lambda$ for $\lambda \in \mathbb{C} \setminus \{0, 1\}$. We clearly have $f_\lambda^n(z) = \lambda^n z$.

If $|\lambda| < 1$ we see that $\lim_{n\to\infty} f_\lambda^n(z) = 0$ for all $z \neq \infty \in \mathbb{C}_\infty$. Similarly to above this implies that

$$\lim_{n\to\infty} f^n(z) = g^{-1}(0) = z_1 \text{ for all } z \neq z_2 \in \mathbb{C}.$$

If $|\lambda| > 1$ we have $\lim_{n\to\infty} f_\lambda^n(z) = \infty$ for all $z \neq 0 \in \mathbb{C}_\infty$ so we see that

$$\lim_{n\to\infty} f^n(z) = g^{-1}(\infty) = z_2 \text{ for all } z \neq z_1 \in \mathbb{C}_\infty.$$

So this means that f progessively moves a point $z \neq z_1, z_2$ away from one of these fixed points and towards the other one. A map with $|\lambda| \neq 1$ and λ is real and positive is called *hyperbolic.* Otherwise it is called *loxodromic.*

Since $tr^2(f) = \lambda + \frac{1}{\lambda} + 2$, we have the following classification of the limiting behaviour of a Möbius transformation, f in terms of the $tr^2(f)$. We formulate this in the following table:

$$\begin{aligned}
&f \text{ is parabolic if and only if } tr^2(f) = 4\\
&f \text{ is hyperbolic if and only if } tr^2(f) > 4\\
&f \text{ is loxodromic if and only if } tr^2(f) < 0 \text{ or } tr^2(f) \notin \mathbb{R}\\
&f \text{ is elliptic if and only if } 0 \le tr^2(f) < 4.
\end{aligned}$$

If f is elliptic f is conjugate to f_λ, $|\lambda| = 1$. For a unit complex number $\lambda \neq 1$ we have $\lambda + \frac{1}{\lambda} \in \mathbb{R}$ and satisfies $-2 \le \lambda + \frac{1}{\lambda} < 2$, which implies that f is elliptic if and only if $0 \le tr^2(f) < 4$.

10 Möbius Transformations of Finite Period

Certain maps $f \in \mathcal{M}(\mathbb{C}_\infty)$ satisfy $f^m(z) = z$. These maps generate finite subgroups of $\mathcal{M}(\mathbb{C}_\infty)$ and we wish to classify which maps have this property. We begin with a definition.

Definition 12. An element $f \in PSL_2(\mathbb{C})$ has period m if m is the smallest integer such that $f^m(z) = z$. If no such integer m exists, f has infinite period.

Theorem 11. *If $f \in PSL_2(\mathbb{C})$ is a non-identity element with finite period, then f is elliptic.*

Proof. First note that f is conjugate to f_λ for some λ, so f^n is conjugate to f_λ^n. Since we assume that f has finite period, say m, we see that f_λ also has finite period, m. We know that $f_1^n(z) = z + n$, so f_1^n has infinite period and we conclude that f is not conjugate to f_1. This implies that $\lambda \neq 1$ and hence $f_\lambda^n(z) = \lambda^n z$. Since f_λ also has period m we see that $\lambda^m z = z$ and hence $\lambda^m = 1$. Hence we see that f is elliptic. ☐

Remark 15. It is not true that all elliptic maps have finite period. If we put $\lambda = e^{i\theta}$ we see that f is elliptic if and only if θ is not an integer multiple of 2π and that it has finite period if and only if θ is a rational (but not integer) multiple of 2π. If θ is irrational, f is an elliptic map which does not have finite period.

11 Rotations of $\mathbb{C}_\infty$

In this section we will study certain Möbius transformations which are rotations. We start by defining rotation in $\mathbb{R}^3$ and then explain what we mean by a rotation of $\mathbb{C}_\infty$. We will show that rotations of $\mathbb{C}_\infty$ are Möbius transformations, and will classify which Möbius transformations are rotations.

Definition 13. A rotation of S^2 is a linear mapwith positive determinant that maps S^2 to itself.

Remark 16. By a theorem of Euler, such a rotation always has an axis that is fixed. By choosing a suitable orthonormal basis, with the vector that is fixed as a basis vector, of $\mathbb{R}^3$, such a map is given by the matrix

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This means that in the orthogonal complement to the vector that is fixed, the map is given by an element of $SO(2)$. A rotation of S^2 extends to a rotation of $\mathbb{R}^3$ by linearity. The set of rotations of $\mathbb{R}^3$ is denoted by $SO(3)$ and forms a group under matrix multiplication. One can show that rotations are conformal (angle preserving) maps of $\mathbb{R}^3$.

We start with the following basic definition.

Definition 14. A map $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is called a rotation of $\mathbb{C}_\infty$ if the map $SP^{-1} \circ f \circ SP\colon S^2 \to S^2$ is a rotation, i.e. an element of $SO(3)$.

Remark 17. The set of rotations of $\mathbb{C}_\infty$ is denoted $Rot(\mathbb{C}_\infty)$ and is a group under composition. Since both stereographic projection and the rotation $SP^{-1} \circ f \circ SP\colon S^2 \to S^2$ are conformal, we see that f has to be conformal as well, i.e. f is an element of $\mathcal{M}(\mathbb{C}_\infty)$ or $PSL_2(\mathbb{C})$, since the Möbius transformations are exactly the conformal maps of $\mathbb{C}_\infty$. In particular, $Rot(\mathbb{C}_\infty) \subseteq \mathcal{M}(\mathbb{C}_\infty)$ is a subgroup, which can be identified with a subgroup of $PSL_2(\mathbb{C})$ in the usual way. In the following we will describe this subgroup concretely.

If $P = (u, v, w) \in S^2$ then we call $\tilde{P} = (-u, -v, -w) \in S^2$ the antipodal point of P. If $z = SP((u, v, w)) \in C_\infty$ the antipodal point of z, $\tilde{z} \in \mathbb{C}_\infty$ is given by $\tilde{z} = SP((-u, -v, -w))$. It follows from the formulas for SP that the antipodal point of z is given by $\frac{-1}{\bar{z}}$. Note that if $f \in Rot(\mathbb{C}_\infty)$ then an antipodal pair $(z, \tilde{z})$ is mapped to an antipodal pair $(f(z), \widetilde{f(z)})$ Since rotating S^2 and then apply the antipodal map is the same as applying the antipodal map and then rotating, we have $f(\tilde{z}) = \widetilde{f(z)}$, which means for a rotation f we have

$$f\left(\frac{-1}{\bar{z}}\right) = \frac{-1}{\overline{f(z)}}.$$

This formula is important for the proof of the next theorem.

Theorem 12. *The group of rotations of $\mathbb{C}_\infty$ is given by*

$$Rot(\mathbb{C}_\infty) = \left\{ \frac{az+b}{-\overline{b}z+\overline{a}} \mid a\overline{a}+b\overline{b}=1 \right\} = PSU_2(\mathbb{C}).$$

Proof. Let $f \in Rot(\mathbb{C}_\infty)$, $f(z) = \frac{az+b}{cz+d}$ and $ad-bc=1$. By the above formula we have

$$\frac{\frac{-1}{\overline{z}}+b}{c\frac{-1}{\overline{z}}+d} = \frac{-1}{\frac{\overline{az}+\overline{b}}{\overline{cz}+\overline{d}}}.$$

which implies

$$\frac{b\overline{z}-a}{d\overline{z}-c} = \frac{-\overline{cz}-\overline{d}}{\overline{az}+\overline{b}}.$$

Comparing these two matrices in $PSL_2(\mathbb{C})$ we get

$$\begin{bmatrix} b & -a \\ d & -c \end{bmatrix} = \pm \begin{bmatrix} -\overline{c} & -\overline{d} \\ \overline{a} & \overline{b} \end{bmatrix}$$

If the factor is -1 we get $b=\overline{c}$ and $a=\overline{-d}$, which means that $1 = ad-bc = -a\overline{a}-b\overline{b} = -(|a|^2+|b|^2) < 0$, which is clearly impossible. If the factor is $+1$ we have $b=-\overline{c}$ and $a=\overline{d}$ and we get $f(z) = \frac{az+b}{-\overline{b}+\overline{a}}$, $a\overline{a}+b\overline{b}=1$. These maps form a subgroup of $PSL_2(\mathbb{C})$, which we denote by $PSU_2(\mathbb{C})$. Hence we have shown that $Rot(\mathbb{C}_\infty) \subseteq PSU_2(\mathbb{C})$.

For the other inclusion, we first noice that if $f(z) = \frac{az+b}{-\overline{b}+\overline{a}}$, $a\overline{a}+b\overline{b}=1$ fixes 0, then $b=0$. This implies that $a\overline{a}=1$ and hence $f(z) = \frac{az}{\overline{a}} = a^2z$, $|a^2|=1$. So f is a rotation by an angle θ, where $a^2 = \mathrm{e}^{i\theta}$. Now let $f(z) = \frac{az+b}{-\overline{b}+\overline{a}}$, $a\overline{a}+b\overline{b}=1$ be any transformation and let $f(z_0)=0$. We let z_0 correspond to $P \in S^2$ and as usual 0 corresponds to S. Then there exists a rotation of S^2, R, with $R(P)=S$. The composit $SP \circ R \circ SP^{-1} = R_1$ takes z_0 to 0, and is by definition a rotation of $\mathbb{C}_\infty$. The map $f \circ R_1^{-1}$ fixes 0 and since $PSU_2(\mathbb{C})$ is a group, is an element of $PSU_2(\mathbb{C})$. By the previous argument, an element of $PSU_2(\mathbb{C})$ that fixes 0 is a rotation, so $f \circ R_1^{-1}(z) = \mathrm{e}^{i\varphi}z = R_2(z)$, for some φ. Now we have $f = R_2 \circ R_1 \in Rot(\mathbb{C}_\infty)$, which shows that $PSU_2(\mathbb{C}) \subseteq Rot(\mathbb{C}_\infty)$. $\square$

Corollary 11.1. *There is a group isomorphism $PSU_2(\mathbb{C}) \cong SO(3)$.*

Proof. Every element in $Rot(\mathbb{C}_\infty)$ corresponds to a rotation of S^2, which extends to a rotation of $\mathbb{R}^3$. This means that $Rot(\mathbb{C}_\infty) \cong SO(3)$. The statement now follows from the previous theorem. $\square$

12 Finite Groups of Möbius Transformations

In this section we want to classify all finite subgroups of $PSL_2(\mathbb{C})$. The presentation will be a bit less detailed then the previous sections. Some algebraic arguments needed to complete the proofs of the theorems are beyond the scope of these notes, and will not be presented.

Recall that f is elliptic if $0 \leq tr^2(f) < 4$ and that by Theorem 11 an element of finite order is elliptic. Hence the finite subgroups of $PSL_2(\mathbb{C})$ must consist of elliptic elements and the identity. Recall that two subgroups $\Gamma_1, \Gamma_2 \in PSL_2(\mathbb{C})$ are conjugate in $PSL_2(\mathbb{C})$ if there exists a $g \in PSL_2(\mathbb{C})$ such that $\Gamma_1 = g\Gamma g^{-1}$. This means that the elements in Γ_1 are simultaneously conjugate (conjugate by the same element) to the elements in Γ_2.

The next theorem says that a subgroup of elliptic transformations together with the identity is conjugate to a subgroup of rotations.

Theorem 13. *Let Γ be a subgroup of $PSL_2(\mathbb{C})$ consisting of elliptic elements together with the identity. Then Γ is conjugate in $PSL_2(\mathbb{C})$ to a subgroup of $PSU_2(\mathbb{C})$.*

Sketch of Proof. Let Γ be a subgroup of $PSL_2(\mathbb{C})$ consisting of elliptic elements together with the identity, and let $\hat{\Gamma}$ be the image of Γ in $SL_2(\mathbb{C})$.

Step 1: Show that by conjugating $\hat{\Gamma}$ we may assume that $\hat{\Gamma}$ contains the element $S = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}$, where $|\lambda| = 1$.

Step 2: Consider an arbitrary element of $\hat{\Gamma}$, $T = \begin{bmatrix} a & b \\ \overline{c} & d \end{bmatrix}$. By calculating $tr^2(ST)$ one shows that $\overline{a} = d$.

Step 3: By calculating $STS^{-1}T^{-1}$ one shows that $b = 0$ if and only if $c = 0$.

Step 4: One shows that there exists an $r \in \mathbb{R} \subseteq \{0\}$ depending on Γ, but not $T \in \Gamma$ such that $c = r\overline{b}$. Put $v = 4\sqrt{|r|}$ and define a transformation $V = \begin{bmatrix} v & 0 \\ 0 & v^{-1} \end{bmatrix} \in SL_2(\mathbb{C})$. Now we have

$$VTV^{-1} = \begin{cases} \begin{bmatrix} a & v^2 b \\ \overline{v^2 b} & \overline{a} \end{bmatrix} & r > 0, \quad (\star) \\ \\ \begin{bmatrix} a & v^2 b \\ -\overline{v^2 b} & \overline{a} \end{bmatrix} & r < 0. \quad (\star\star) \end{cases}$$

This implies that by replacing $\hat{\Gamma}$ by $V\hat{\Gamma}V^{-1}$ we may assume that the elements in Γ

are of the form ($\star$) or ($\star\star$). Since $SU(2) = \left\{ \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} \mid a\overline{a} + b\overline{b} = 1 \right\}$ it is enough to show that the elements of $\hat{\Gamma}$ are of the form ($\star\star$). We also see that $VSV^{-1} = S$, even after replacing $\hat{\Gamma}$ by $V\hat{\Gamma}V^{-1}$, $\hat{\Gamma}$ still contains S.

Step 5: Consider an element $= \begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix}$ of type ($\star$) in $\hat{\Gamma}$. Since $S, T \in \hat{\Gamma}$ the product $STS^{-1}T^{-1}$ is an element of $\hat{\Gamma}$. One now checks that $tr^2(STS^{-1}T^{-1}) > 4$, so $STS^{-1}T^{-1}$ is neither the identity, nor elliptic. This is a contradiction and hence $\hat{\Gamma}$ consists of elements of type ($\star\star$), which means that Γ is conjugate to a subgroup of $PSU_2(\mathbb{C})$. $\square$

Corollary 12.1. *Every finite group of Möbius transformations is conjugate to a group of rotations of* $\mathbb{C}_\infty$.

Proof. As previously noted, by Theorem 11 an element of finite order is elliptic. Hence the finite subgroups of $PSL_2(\mathbb{C})$ must consist of elliptic transformations and the identity. By the previous theorem, such a subgroup is conjugate in $PSL_2(\mathbb{C})$ to a subgroup of rotations. $\square$

The proof of the following theorem relies on some results in group theorey, and will be omitted.

Theorem 14. *Let Γ be a finite group of rotations in $\mathbb{C}_\infty$. Then one of the following holds.*

1. *Γ is cyclic,*
2. *Γ is dihedral,*
3. *Γ is the symmetry group of a regular tetrahedron (A_4), octahedron (S_4) or icosahedron (A_5).*

One can show that two finite subgroups in $PSL_2(\mathbb{C})$ are conjugate if and only if they are isomorphic. Combining the previous two theorems yields the following classification result.

Corollary 12.2. *Every finite subgroup of $PSL_2(\mathbb{C})$ is cyclic, dihedral, or isomorphic to A_4, S_4 or A_5.*

Proof. By Theorem 13 and the remark before it, every finite subgroup of rotations, i.e. a subgroup of $PSU_2(\mathbb{C})$. The corollary now follows from Theorem 14 and the fact that two subgroups of $PSL_2(\mathbb{C})$ are conjugate if and only if they are isomorphic. $\square$

13 Bibliography

1. Churchill, Ruel V. and Brown, James Ward
   Complex variables and applications
   McGraw-Hill Book Co. New York
   1984.

2. Conway, John B.
   Functions of one complex variable
   Graduate Texts in Mathematics
   Springer-Verlag, New York
   1978.

3. Jones, Gareth A. and Singerman, David
   Complex functions - An algebraic and geometric viewpoint
   Cambridge University Press, Cambridge
   1987.

4. Saff, Edward B. and Snider, Arthur David
   Fundamentals of complex analysis with applications to engineering, Science, and Mathematics
   Prentice Hall.
   2003.

Comments and corrections are much appreciated.
Please send them to: johno2211@googlemail.com.

14 Problems

Problem 1. Show that a line $ax + by = c/2$ can be written as $\overline{A}z + A\overline{z} = c$, c real. Find the equation for the line through i and $1 + 2i$.

Problem 2. Find a formula for the reflection in the line $\overline{A}z + A\overline{z} = c$, $|A| = 1$. What is the formula for the reflection in the line $x + y = 1$?

Problem 3. Find the formula for the circle $\{z \in \mathbb{C} \mid |z - C| = r\}$ in terms of x and y, $z = x + iy$. Find the equation for the circle of radius 2 and center i.

Problem 4. Write down the formula for the direct affine transformation given by:

1. Translation in the direction $(2, -3)$.
2. Rotation about $(0, 1)$ through $\pi/4$.

Problem 5. Let SP be the stereographic projection from the north pole. Let $P = (u, v, w)$ and $z = x + iy$. Let $SP(P) = z$. Derive the formulas for u, v, w in terms of x, y (that is, fill in the details in the proof of Proposition 3.1).

Problem 6. Show that the stereographic projection preserves angles. That is, for r and s lines in $\mathbb{C}$ inclined at an angle θ at the origin (i.e. $p = 0$ in the formulas in the proof of Proposition 3.2), find the equations for the line on S^2 and show that the angle between the tangent vectors at $t = 0$ is θ.

Problem 7. A fixed point for $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a point z such that $f(z) = z$. Find the fixed points (if any) of a dilation, translation, a general direct affine transformation and the inversion. Remember to check if infinity is a fixed point!

Problem 8. Find the image of the horizontal line $y = c$, both for $c = 0$ and $c \neq 0$, under the inversion map.

Problem 9. If the unit circle is oriented counter-clockwise, determine the orientation of the image of the unit circle under the inversion map. What is the image of the center of the circle?

Problem 10. Determine the image of the disk $\{z \in \mathbb{C} \mid |z| < 1\}$ under the inversion map. Interpret this in terms of inversion on S^2.

Problem 11. Find a linear transformation that takes the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ onto the circle $\{w \in \mathbb{C} \mid |w - 5| = 3\}$ and maps the point $z = i$ to the point $w = 2$. Note the word linear!

Problem 12. Show that a Möbius transformation f can have at most two fixed points in the complex plane unless f is the identity.

Problem 13. Discuss the image of the circle $|z-2|=1$ and its interior under the maps

1. $f(z) = z - 2i$
2. $f(z) = 3iz$
3. $f(z) = \frac{z-2}{z-1}$
4. $f(z) = \frac{z-4}{z-3}$
5. $f(z) = \frac{1}{z}$.

Problem 14. Find a Möbius transformation that maps $\{z \in \mathbb{C} \mid |z| < 1\}$ onto the right half plane $\{z \in \mathbb{C} \mid \Re(z) > 0\}$ and takes $-i$ to the origin.

Problem 15. Find the Möbius transformation that maps $(0, 1, \infty)$ to the points $(0, i, \infty)$, $(0, 1, 2)$, $(-i, \infty, 1)$ and $(-1, \infty, 1)$.

Problem 16. What is the image of the third quadrant under the map $f(z) = \frac{z+i}{z-i}$?

Problem 17. What is the image of the sector $\{z \in \mathbb{C} \mid -\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}\}$ under the map $f(z) = \frac{z}{z-1}$?

Problem 18. Show that a Möbius transformation has ∞ as its only fixed point if and only if it is a translation, but not the identity. Show that a Möbius transformation has 0 and ∞ as its only fixed point if and only if it is a dilation, but not the identity.

Problem 19. Let f be a Möbius transformation with fixed points z_1 and z_2. If g is another Möbius transformation, show that $g^{-1}fg$ has fixed points $g^{-1}(z_1)$ and $g^{-1}(z_2)$.

Problem 20. Two Möbius transformations f and g are said to commute if $fg = gf$. Let f be a Möbius transformation not equal to the identity. Show that a Möbius transformation g commutes with f if g and f have the same fixed points. Hint: Use the statements in the previous problems.

Problem 21. Show that the set of elliptic transformations in $\mathcal{M}(\mathbb{C}_\infty)$ together with the identity form a subgroup of $\mathcal{M}(\mathbb{C}_\infty)$ under composition of maps.

Problem 22. Consider the Möbius transformation $f(z) = \frac{z-2}{z-1}$.

1. Determine the fixed points of f.
2. Calculate $tr^2(f)$ and the multiplier of f.
3. Find a representative (not f itself) of the conjugacy class of f.
4. Describe the limiting behavior of f, $\lim_{n\to\infty} f^n(z)$ for all $z \in \mathbb{C}_\infty$.

Problem 23. Consider the Möbius transformation $f(z) = \frac{iz}{z-i}$.

1. Determine the fixed points of f.
2. Calculate $tr^2(f)$ and the multiplier of f.
3. Find a representative (not f itself) of the conjugacy class of f.
4. Describe the limiting behavior of f, $\lim_{n\to\infty} f^n(z)$ for all $z \in \mathbb{C}_\infty$.

Problem 24. By using the formulas for stereographic projection, check that the antipodal point to $z \in \mathbb{C}_\infty$ is given by $\frac{-1}{\bar{z}}$.

Problem 25. Show that the set of rotations of $\mathbb{C}_\infty$ form a group under composition of maps.

Problem 26. Show that $PSU_2(\mathbb{C})$ form a subgroup of $PSL_2(\mathbb{C})$.

Problem 27. Let (z_0, z_1, z_2, z_3) be four distinct points in $\mathbb{C}_\infty$. Show that there are precisely two values of k such that (z_0, z_1, z_2, z_3) can be mapped to $(1, -1, k, -k)$ by a Möbius transformation.

Problem 28. Show that a rotation of $\mathbb{C}_\infty$ is represented by an elliptic element $PSL_2(\mathbb{C})$.

Problem 29. Show that an elliptic element in $PSL_2(\mathbb{C})$ whose fixed points are antipodal points in $\mathbb{C}_\infty$ is a rotation of $\mathbb{C}_\infty$.

Problem 30. Let f and g be Möbius transformations with a common fixed point z_0. Show that the Möbius transformation $f \circ g \circ f^{-1} \circ g^{-1}$ is either parabolic or the identity. (Hint: Assume $z_0 = \infty$. Why is this method valid?)