Topological Investigations of Positively Curved Manifolds

Master Thesis in Mathematics

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July 2004

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Contents

Introduction

In this thesis we will discuss the connectedness principle and its consequences in the geometry of positive sectional curvature. The results are in nature topological and the Riemannian geometric assumption enters the proofs via a Morse Theory argument, a so called “Synge type argument”.

Let M be a complete manifold. In [G II] Grove studied geodesics satisfying certain boundary conditions. Concretely, let $N \subset M \times M$ be a submanifold. A geodesic $\gamma\colon [0,1] \longrightarrow M$ is called an N-geodesic if

$$(\gamma(0), \gamma(1)) \in N \text{ and } (\dot\gamma(0), -\dot\gamma(1)) \text{ is normal to } N.$$

Here $\dot\gamma(t)$ denotes the tangent vector of γ at t and the manifold $M \times M$ is endowed with the product metric. Let also Δ be the diagonal in $M \times M$.

Using critical point theory on Hilbert manifolds, Grove proved the following results.

Lemma. *Let $\Lambda_N(M) = \{\sigma \in L_1^2(I, M) \mid (\sigma(0), \sigma(1)) \in N\}$. If there are no nontrivial N-geodesics on M then the inclusion*

$$e\colon N \cap \Delta \longrightarrow \Lambda_N(M), \quad e(x,x)(t) = x \quad \forall t \in I$$

is a homotopy equivalence.

Using this lemma and some elementary homotopy theory Grove proved the following theorem.

Theorem A. *Let $p_i\colon N \longrightarrow M$ be projection on the ith factor. If there are no nontrivial N-geodesics on M then there is an exact sequence of homotopy groups*

$$\longrightarrow \pi_{i+1}(N) \xrightarrow{(p_1)_* - (p_2)_*} \pi_{i+1}(M) \longrightarrow \pi_i(N \cap \Delta) \xrightarrow{i_*} \pi_i(N) \longrightarrow$$

for $i \geq 0$.

For $i \geq 1$ exactness at $\pi_{i-1}(N \cap \Delta)$ implies that

$$\forall\ [k] \in \pi_i(M)\ \exists\ [g] \in \pi_i(N)\ \exists [h] \in \pi_i(N, N \cap \Delta) \quad \text{such that}$$
$$[k] = ((p_1)_*([h]) - (p_2)_*([h])) - ((p_1)_*([g]) - (p_2)_*([g])).$$

If $i = 1$ read the formula multiplicative.

The main theorem of Chapter 3 is clearly inspired by this theorem. The rest of [G II] is devoted to proving results on $V_1 - V_2$-connecting geodesics i.e. geodesics connecting the compact connected submanifolds V_1 and V_2 of M. These results are all corollaries of the above theorem.

Contents

This thesis is divided into four parts. For each of these we will now summarize the most important results.

In the first chapter we first collect some basic definitions and results from algebraic topology and Riemannian geometry which will be used throughout the text. The connection between the Riemannian geometric properties and the topological properties of the manifolds is established via a Morse Theory argument. A complete account of this theory is given in Milnor's beautiful book [M]. We will however use some of the results in a slightly more general version. For this reason, we will generalize the results to that set-up.

In the second chapter we will use the Morse Theory from Chapter 1 to prove some classical theorems by Frankel and Synge, see [F] and [S].

Theorem B. *Assume that M^m has positive sectional curvature. Let V^s and W^r be two compact totally geodesic embedded submanifolds. If $r + s \geq m$ then $V \cap W \neq \emptyset$.*

Theorem C. *Assume that M has positive sectional curvature.*

(1) If M is even dimensional and orientable, then M is simply connected.

(2) If M is odd dimensional, then M is orientable.

A more serious application of Morse Theory is the following recent connectedness theorem by Wilking, see [W I]. This theorem is the central result of this thesis.

Theorem D. *Assume that M^m has positive sectional curvature.*

(1) Suppose $N^{m-k} \subset M^m$ is a compact totally geodesic embedded submanifold. Then the inclusion map $i\colon N^{m-k} \longrightarrow M^m$ is $(m-2k+1)$-connected.

(2) Suppose $N_1^{m-k_1}, N_2^{m-k_2} \subset M^m$ are two compact totally geodesic embedded submanifolds with $k_1 \leq k_2$ and $k_1 + k_2 \leq m$. Then the intersection is a compact totally geodesic embedded submanifold as well and the inclusion $i\colon N_1^{m-k_1} \cap N_2^{m-k_2} \longrightarrow N_2^{m-k_2}$ is $(m-k_1-k_2)$-connected.

We will briefly indicate how the proof of (1) goes. One starts by considering the space $P(M,N,N)$ of piecewise smooth curves starting and ending in N. The energy of such a curve is given by $\mathrm{E}(\gamma) = \int_0^1 |\dot{\gamma}|^2 dt$. Notice that N embeds naturally as $\mathrm{E}^{-1}(0)$. The critical points of E are the smooth geodesics starting and ending orthogonally to N.

Let $P_{<c}(M,N,N) = \mathrm{E}^{-1}([0,c))$. For a $c > 0$ and a sufficiently fine partition $0 = t_0 < t_1 < \cdots < t_k = 1$ the space

$$B_c^k(M,N,N) =$$
$$\{\gamma \in P(M,N,N) \mid \gamma_{|[t_{i-1},t_i]} \text{ minimal geodesic for all } i = 1,\dots,k\} \cap P_{<c}(M,N,N)$$

has the structure of a finite dimensional smooth manifold and the map

$$\mathrm{E}_{|B_c^k(M,N,N)} : B_c^k(M,N,N) \longrightarrow [0,c)$$

is smooth and proper. Furthermore, the space $B_c^k(M,N,N)$ is homotopy equivalent to the space $P_{<c}(M,N,N)$.

Since the number of linearly independent smooth and parallel vector fields along a nontrivial critical point of E is $m-2k+1$ it follows that the index of a nontrivial critical point is at least $m-2k+1$. The facts that $\mathrm{E}^{-1}(0) = N$, $\mathrm{E}_{|B_c^k(M,N,N)}$ is smooth and proper and the index of a nontrivial critical point is at least $m-2k+1$ imply by Morse Theory that $\pi_i(B_c^k(M,N,N),N) = 0$ for $i \leq m-2k$. Since $B_c^k(M,N,N)$ is homotopy equivalent to the space $P_{<c}(M,N,N)$ it follows from a homotopy direct limit argument that $\pi_i(P(M,N,N),N) = 0$ for $i \leq m-2k$. Using the isomorphism $\pi_i(P(M,N,N),N) \cong \pi_{i+1}(M,N)$ one sees that the inclusion $i\colon N^{m-k} \longrightarrow M^m$ is $(m-2k+1)$-connected.

It is worth noting that Frankel's Theorem is an immediate consequence of Wilking's Theorem, since $\pi_0(\emptyset) = \emptyset$ and the map $i\colon N_1^{m-k_1} \cap N_2^{m-k_2} \longrightarrow N_2^{m-k_2}$ is $(m-k_1-k_2)$-connected.

Poincare Duality implies that this theorem has some very strong consequences for the cohomology ring of M.

Theorem E. *Let M^m be a compact oriented manifold and let N^{m-k} be an embedded compact oriented submanifold. Suppose the inclusion $i\colon N^{m-k} \longrightarrow M^m$ is $(m-k-l)$-connected and $m-k-2l > 0$. Let $[N] \in H_{m-k}(M;\mathbb{Z})$ be the image of the fundamental class of N in $H_*(M;\mathbb{Z})$ and let $e \in H^k(M;\mathbb{Z})$ be its Poincare dual. Then the homomorphism*

$$\cup e\colon H^i(M;\mathbb{Z}) \longrightarrow H^{i+k}(M;\mathbb{Z})$$

is surjective for $l \leq i < m-k-l$ and injective for $l < i \leq m-k-l$.

In the third chapter we examine the twisted path space and use this construction to derive a new connectedness theorem. In particular, we prove a generalization of Wilking's theorem of Chapter 2. The results of this chapter are due to Fang, Mendonca and Rong, see [FMR].

Let $f\colon N \longrightarrow M \times M$ be an isometric immersion. Let $P(M)$ be the space of piecewise smooth curves in M. Consider the fibration $ev_{0,1}\colon P(M) \longrightarrow M \times M$ given by $\gamma(t) \longmapsto (\gamma(0), \gamma(1))$. The twisted path space is $P(M, f) = f^*(P(M))$, the pull-back fibration over N. We define the energy of a point in $P(M, f)$ to be $\mathrm{E}(x, \gamma) = \int_0^1 |\dot{\gamma}|^2 dt$. Let $f^{-1}(\Delta)$ be the inverse image of the diagonal in $M \times M$. This is identified with $\mathrm{E}^{-1}(0)$.

We start by showing that the Morse Theory approach developed in Chapter 1 also works when we consider the twisted path space. Using this approach to Morse Theory, we prove the following theorem.

Theorem F. *Let M and N be compact and path connected Riemannian manifolds and let $f\colon N \longrightarrow M \times M$ be an isometric immersion. Assume that every nontrivial critical point of $\mathrm{E} : P(M, f) \longrightarrow \mathbb{R}$ has index $\geq \lambda$.*

(1) If $\lambda \geq 1$ then $f^{-1}(\Delta) \neq \emptyset$

(2) If $\lambda \geq 2$ and M is simply connected then $f^{-1}(\Delta)$ is path connected.

If, in addition, f is either minimal or $N = N_i \times N_i$ $i = 1, 2$ and $f = f_i \times f_i$ where f_i is an embedding then

(3) $\pi_i(P(M, f), f^{-1}(\Delta)) = 0$ for all $i < \lambda$.

(4) there is an exact sequence of homotopy groups

$$\longrightarrow \pi_i(f^{-1}(\Delta)) \longrightarrow \pi_i(N) \xrightarrow{\partial} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta)) \longrightarrow$$

where $i < \lambda$. ∂ is given by $(p_1 f)_* - (p_2 f)_*$ where p_1, p_2 are the projections of $M \times M$ onto the first and second factor, respectively.

The proof of (3) is similar to the proof of Wilking's theorem and (4) follows from (3) by elementary homotopy theory.

The asymptotic index $\nu_f(x)$ of an isometric immersion $f\colon N \longrightarrow M \times M$ at a point $x \in N$ is the maximal dimension of a subspace N_x of $T_x N$ such that the second fundamental form satisfies $II_{|N_x} = 0$. If N is compact the asymptotic index of f is given by $\nu_f = \min_{x \in N} \nu_f(x)$. One can estimate the index of a nontrivial critical point of E in $P(M, f)$ in terms of the asymptotic index of f.

Proposition. *Assume that M and N are compact and path connected Riemannian manifolds. Assume that M^m has positive lth Ricci curvature and let $f\colon N \longrightarrow M \times M$ be an isometric immersion. Let (x, γ) be a critical point of E in $P(M, f)$ and let λ be the index of (x, γ) as a critical point. Then*

(1) $\lambda \geq \nu_f - m - l + 1$.

(2) If $f = f_1 \times f_2 \colon N_1 \times N_2 \longrightarrow M \times M$ *where* $f_i \colon N_i \longrightarrow M$ *is an immersion then* $\lambda \geq \nu_f - m - l + 2$*.*

Using this one can prove the following generalization of Wilking's theorem.

Theorem G. *Assume that* M^m *has positive sectional curvature. Let* N_j*,* $j = 1, 2$ *be embedded compact path connected submanifolds with asymptotic index* ν_j*. Then*

(1) the inclusion $i_j \colon N_j \longrightarrow M$ *is* $(2\nu_j - m + 1)$*-connected.*

(2) if N_j *are both minimal (the inclusion is minimal),* $\nu_1 + \nu_2 \geq m$ *and* $\nu_2 \geq \nu_1$ *then* $i \colon N_1 \cap N_2 \longrightarrow N_1$ *is* $(\nu_1 + \nu_2 - m)$*-connected.*

Notice that Wilking's theorem follows from this one since a totally geodesic embedding $i \colon N \longrightarrow M$ has asymptotic index $\nu_j = \dim(N)$.

In the fourth chapter we will investigate the homotopy theoretical and cohomological properties of manifolds with totally geodesic submanifolds. The results of this chapter are due to Wilking, see [W I]. The results are all derived from Wilking's two theorems in Chapter 2 using Poincare Duality and some algebraic topology. The main result is the following.

Theorem H. *Suppose* M^m *is simply connected, oriented and has positive sectional curvature.*

(1) If m *is odd and* M *contains one embedded totally geodesic compact oriented submanifold* N *of codimension* 2 *then* M *is homotopy equivalent to a sphere (and hence homeomorphic if* $m > 3$*).*

(2) If $m \geq 4$ *is even and* M *contains a compact embedded totally geodesic oriented submanifold* N *of codimension* 2 *and a compact embedded totally geodesic oriented submanifold* N' *of codimension* $< m/2$ *with* $N' \cap N$ *transverse, then* M *is homotopy equivalent to a sphere or to a complex projective space.*

(3) Suppose $m \equiv 0 \pmod 4$ *or* $m \equiv 1 \pmod 4$ *and* $m \geq 13$*. If* M *contains a compact embedded totally geodesic oriented submanifold* N_1 *of codimension* 4 *and a compact embedded totally geodesic oriented submanifold* N_2 *of codimension* $\leq m/2 - 3$ *such that* N_1 *and* N_2 *intersect transversely, then* M *has the cohomology ring of either* S^m*,* $\mathbb{CP}^{m/2}$ *or* $\mathbb{HP}^{m/4}$*.*

Acknowledgement

I would like to thank my supervisor, Karsten Grove, for interesting and inspiring discussions and for suggesting this fascinating topic in the first place! Jørgen Tornehave supervised this project during the Spring 2004, and I would like to thank him for many inspiring discussions and useful advice. I would also like to thank Marcel Bökstedt for his enthusiasm and interest. Many thanks to Rune Ljungmann for proofreading the thesis.

Chapter 1

Prerequisite

We start by presenting some definitions and facts from differential geometry and algebraic topology which we will need throughout the text. We also present a slightly more general version of some of the results presented in [M, Chapter 12 - Chapter 16]. These results from Morse Theory will be used several times at vital points in the text.

Throughout this thesis we let M be a complete Riemannian manifold. For simplicity we will assume throughout the text, that the manifolds considered are without boundary. All curves are parametrized by $[0, 1]$.

1.1 Differential Geometry and Algebraic Topology

As these results are supposed to be “well-known”, proofs will only be given if they are shorter than a reference!

Riemannian Geometry

The results and definitions of this section are also valid if M is not complete.

Assume that (N, h) and (M, g) are Riemannian manifolds with metrics h and g respectively. Let $\nabla^{M,g}$ be the Riemannian connection on M corresponding to the metric g and let $\nabla^{N,h}$ be the Riemannian connection on N corresponding to the metric h. Let $f\colon (N, h) \longrightarrow (M, g)$ be an isometric immersion, that is an immersion which for all $x \in N$ satisfies $\langle v, w\rangle = \langle f_*(v), f_*(w)\rangle$, for $v, w \in T_xN$.

Definition 1.1. The second fundamental form $II\colon TN \otimes TN \longrightarrow TN^{\perp}$ of f is pointwise given by

$$(\nabla^{M,g}_X(Y))_x = (\nabla^{N,h}_X(Y))_x + (II(X, Y))_x,$$

where X, Y are vector fields on N and $(\nabla^{N,h}_X(Y))_x$, $(II(X, Y))_x$ are the tangential and normal component of the vector $(\nabla^{M,g}_X(Y))_x$ at the point x.

By dualizing we get a map $II'\colon TN \otimes TN^{\perp} \longrightarrow TN$ defined pointwise by

$$\langle II'(U,X),V\rangle = -\langle II(U,V),X\rangle,$$

for $U,V \in T_xN$ and $X \in T_xN^{\perp}$, see e.g. [KN II, Chapter 3].

Definition 1.2. An isometric immersion f is totally geodesic if $II \equiv 0$

Definition 1.3. The asymptotic index $\nu_f(x)$ of f at a point $x \in N$ is the maximal dimension of a subspace N_x of T_xN such that $II_{|N_x} = 0$. If N is compact, the asymptotic index of f is $\nu_f = \min_{x\in N} \nu_f(x)$.

Remark. An isometric immersion f is totally geodesic if and only if $\nu_f = \dim(N)$ △

Definition 1.4. A Riemannian manifold M^m with curvature tensor R has positive kth Ricci curvature if for $v_1,\dots,v_{k+1}$ orthonormal vectors in T_pM the number

$$\sum_{j=1}^{k+1} \langle R(v_j,v)v,v_j\rangle > 0,$$

for all $v \neq 0 \in \operatorname{span}\{v_1,\dots,v_{k+1}\}$.

Remark. Positive $(m-1)$'st Ricci curvature is positive Ricci curvature.

Positive first Ricci curvature is positive sectional curvature. Let v_1, v_2 be orthonormal vectors in T_pM. Consider $\langle R(v_1,v)v,v_1\rangle + \langle R(v_2,v)v,v_2\rangle$ for $v \in \operatorname{span}\{v_1,v_2\}$. Put $v = c_1v_1 + c_2v_2$ then by linearity and antisymmetry of R we get

$$\begin{aligned}\langle R(v_1,v)v,v_1\rangle + \langle R(v_2,v)v,v_2\rangle &= c_2^2\langle R(v_1,v_2)v_2,v_1\rangle + c_1^2\langle R(v_1,v_2)v_2,v_1\rangle \\ &= (c_1^2+c_2^2)\langle R(v_1,v_2)v_2,v_1\rangle.\end{aligned}$$

It follows that M has positive first Ricci curvature if and only if M has positive sectional curvature. △

Definition 1.5. The mean curvature vector of f at $x \in N$ is the normal vector H_x defined by $H_x = \operatorname{trace} II_x = \sum_{i=1}^{\dim N} II(X_i,X_i)$, where $\{X_i\}$ is an orthonormal basis of T_xN.

Definition 1.6. An isometric immersion f is minimal if $H_x \equiv 0$ for all $x \in N$.

Algebraic Topology

We will also collect a few definitions and results from algebraic topology. Let R be a ring with unit.

Definition 1.7.

(1) A local orientation of a manifold M at $x \in M$ is a choice of generator for the infinite cyclic group $H_n(M, M \setminus \{x\}; \mathbb{Z})$.

(2) An orientation is a function $x \longmapsto \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M, M \setminus \{x\}; \mathbb{Z})$ satisfying the following condition: Each $x \in M$ has a neighbourhood $U \subseteq M$ homeomorphic to $\mathbb{R}^n$ containing an open ball B of finite radius such that all the local orientations μ_y at $y \in B$ are the image of one generator $\mu_B \in H_n(M, M \setminus B; \mathbb{Z}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B; \mathbb{Z})$ under the natural map $H_n(M, M \setminus B; \mathbb{Z}) \longrightarrow H_n(M, M \setminus \{y\}; \mathbb{Z})$. If an orientation exists, M is called orientable.

(3) An R-orientation is a function $x \longmapsto \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M, M \setminus \{x\}; R)$ satisfying the condition of (2). If an R-orientation exists M is called R-orientable.

One can also define an orientable smooth manifold to be a smooth manifold which possesses an oriented smooth atlas.

Recall that every manifold has a two sheeted covering $\overline{M}$ given by

$$\overline{M} = \{\mu_x \mid x \in M,\ \mu_x \text{ a local orientation of } M \text{ at } x\}.$$

The map $\mu_x \longmapsto x$ is the covering projection.

Proposition 1.8. *If M is connected then M is orientable if and only if $\overline{M}$ has two components. In particular M is orientable if it is simply connected or more generally if $\pi_1(M)$ contains no subgroup of index two.*

Proof: [H, Proposition 3.25]. □

To define the fundamental class of a manifold we need the following theorem.

Theorem 1.9. *Let M be a compact and connected n-manifold. Then*

(1) If M is R-orientable the map $H_n(M; R) \longrightarrow H_n(M, M \setminus \{x\}; R) \cong R$ is an isomorphism for all $x \in M$.

(2) if M is not R-orientable the map $H_n(M; R) \longrightarrow H_n(M, M \setminus \{x\}; R) \cong R$ is injective with image $\{r \in R \mid 2r = 0\}$ for all $x \in M$.

Proof: [H, Theorem 3.26]. □

In view of the theorem we make the following definition.

Definition 1.10. An element of $H_n(M;R)$ whose image in $H_n(M, M\setminus\{x\};R)$ is a generator for all $x \in M$ is called a fundamental class of M and is denoted $[M]$.

An essential ingredient in the proof of Wilking's cohomology theorem is the Poincare Duality Theorem.

Theorem 1.11. *If M is a compact R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$ then the map $PD\colon H^k(M;R) \longrightarrow H_{n-k}(M;R)$ defined by $PD(\alpha) = [M] \cap \alpha$ is an isomorphism for all k.*

Proof: [H, Theorem 3.30]. □

Corollary 1.12. *Let M be a compact and R-orientable manifold. If R is a field or $R = \mathbb{Z}$ and the torsion of $H^*(M, \mathbb{Z})$ is factored out, the bilinear pairing*

$$H^k(M;R) \times H^{n-k}(M;R) \longrightarrow R$$

given by $(\varphi, \psi) \longmapsto (\varphi \cup \psi)[M]$ is nonsingular.

Proof: [H, Proposition 3.38]. □

Remark. Let $\mathbb{F}$ be a field. The pairing defined in Corollary 1.12 gives an isomorphism $H^i(M;\mathbb{F}) \cong H^{n-i}(M;\mathbb{F})^*$. Since M is compact all the cohomology groups are finite dimensional vector spaces over $\mathbb{F}$ there is an (non-canonical) isomorphism $H^{n-i}(M;\mathbb{F})^* \cong H^{n-i}(M;\mathbb{F})$. The isomorphism $H^i(M;\mathbb{F}) \cong H^{n-i}(M;\mathbb{F})$ will also be called Poincare Duality. △

We shall also need the concept of a homotopy direct limit of a directed system of spaces.

Consider a sequence of topological spaces $X_0 \subset X_1 \subset X_2 \subset \cdots$ with union X. Define a space (Milnor's telescope) by

$$X_\Sigma = X_0 \times [0,1] \cup X_1 \times [1,2] \cup X_2 \times [2,3] \cup \ldots .$$

Topologize this set as a subset of $X \times \mathbb{R}$.

Definition 1.13. The space X is the homotopy direct limit of the sequence $\{X_i\}$ if the projection map $p\colon X_\Sigma \longrightarrow X$ defined by $p(x,\tau) = x$ is a homotopy equivalence.

The reason why homotopy direct limits are so useful is the following theorem.

Theorem 1.14. *Let X be the homotopy direct limit of $\{X_i\}$ and let Y be the homotopy direct limit of $\{Y_i\}$. Let $f\colon X \longrightarrow Y$ be a map which carries each X_i into Y_i by a homotopy equivalence. Then f itself is a homotopy equivalence.*

Proof: [M, Theorem A]. □

We shall also need the following definition.

Definition 1.15. A map $f\colon X \longrightarrow Y$ is called n-connected if the induced map[1]

$$f_*\colon \pi_i(X) \longrightarrow \pi_i(Y)$$

is an isomorphism for $i < n$ (bijection for $i = 0$) and a surjection for $i = n$.

We will also need the Euler class and a basic property of that characteristic class. We follow [MS] closely.

Denote by ξ the vector bundle $F \longrightarrow E \longrightarrow B$ of dimension $n > 0$ and let $p\colon E \longrightarrow B$ be the projection. Let E_0 be the set of non-zero elements in E and let $F_0 = F \cap E_0$.

Definition 1.16. An orientation of ξ is a function which assigns a orientation to each fibre F of ξ, subject to the following local compatibility condition. For every $b_0 \in B$ there should exists a local coordinate system (U, h), with $b_0 \in U$ and $h\colon U \times \mathbb{R}^n \longrightarrow p^{-1}(U)$ so that for each fibre $F = p^{-1}(b)$ over U the homomorphism $\mathbb{R}^n \longrightarrow F$ given by $x \longmapsto h(b, x)$ is orientation preserving.

One can interpret this in terms of cohomology as follows. To each fibre F there is assigned a preferred generator $u_F \in H^n(F, F_0; \mathbb{Z}) \cong \mathbb{Z}$. The local compatibility condition implies that for every point $b \in B$ there exists a neighbourhood U and a cohomology class $u \in H^n(p^{-1}(U), p^{-1}(U)_0; \mathbb{Z})$ such that for every fibre over U the restriction $u_{|(F,F_0)} \in H^n(F, F_0; \mathbb{Z})$ is equal to u_F.

Now we can state the Thom Isomorphism Theorem.

Theorem 1.17. *Let ξ be an oriented vector bundle of dimension $n > 0$. Then the cohomology group $H^i(E, E_0; \mathbb{Z})$ is zero for $i < n$, and $H^n(E, E_0; \mathbb{Z})$ contains a unique cohomology class u (the fundamental class) whose restriction $u_{|(F,F_0)} \in H^n(F, F_0; \mathbb{Z})$ is equal to the preferred generator u_F for every fibre F of ξ. Furthermore, the correspondence $y \longmapsto y \cup u$ maps $H^k(E; \mathbb{Z})$ isomorphically onto $H^{k+n}(E, E_0; \mathbb{Z})$ for every integer k.*

Proof: See [MS, Theorem 10.4]. □

[1] The choice of basepoint will be suppressed in the following. We trust the reader to imagine what the expressions would look like, if a basepoint was written explicitly.

Remark. It follows that $H^{k+n}(E, E_0; \mathbb{Z})$ is isomorphic to $H^k(B; \mathbb{Z})$. This isomorphism is given by the map $\Phi\colon H^k(B; \mathbb{Z}) \longrightarrow H^{k+n}(E, E_0; \mathbb{Z})$, $\Phi(x) = p^*(x) \cup u$, which is called the Thom Isomorphism. △

We can now define the Euler class. Given a vector bundle of dimension n, consider the restriction homomorphism $(E, \emptyset) \subset (E, E_0)$. This homomorphism gives rise to a homomorphism $H^*(E, E_0; \mathbb{Z}) \longrightarrow H^*(E; \mathbb{Z})$ which we denote by $y \longmapsto y_{|E}$. If we apply this homomorphism to the fundamental class of $u \in H^n(E, E_0; \mathbb{Z})$ we obtain a class $u_{|E} \in H^n(E; \mathbb{Z})$. Also, the projection induces a canonical isomorphism $p^*\colon H^n(B; \mathbb{Z}) \longrightarrow H^n(E; \mathbb{Z})$.

Definition 1.18. The Euler class of an oriented n-dimensional vector bundle ξ is the cohomology class

$$e(\xi) \in H^n(B; \mathbb{Z})$$

which corresponds to $u_{|E}$ under the isomorphism $p^*\colon H^n(B; \mathbb{Z}) \longrightarrow H^n(E; \mathbb{Z})$.

Proposition 1.19. *If the fibre dimension of ξ is odd, then $e(\xi) + e(\xi) = 0$*

Proof: Since the Euler class changes sign if the orientation is reversed and any odd dimensional vector bundle possesses an orientation reversing automorphism $(b, v) \longmapsto (b, -v)$ the result follows. □

Remark. A fundamental class with $\mathbb{Z}$-coefficients gives rise to a fundamental class for any other commutative coefficient ring R under the homomorphism $H^n(E, E_0; \mathbb{Z}) \longrightarrow H^n(E, E_0; R)$ induced by the homomorphism $\mathbb{Z} \longrightarrow R$ defined by $1 \longmapsto 1_R$. One should also note that in an arbitrary ring a generator is an element with a multiplicative inverse. Similarly for the Euler class. △

1.2 Morse Theory

In this section we will give some results and definitions from Morse Theory. This is by no means meant as a complete account, but merely as a presentation of some useful facts and definitions. The definitions, which are left out, can be found in [M]. Most of the proofs are nearly identical to those given in [M], so we only indicate the necessary changes. First we introduce some notation. Let V and W be compact embedded submanifolds of M.

Notation. We let

i) $P(M) = \{\gamma \mid \gamma \text{ piecewise smooth curve}\}$.

ii) $P(M, V, W) = \{\gamma \mid \gamma \text{ piecewise smooth curve } \gamma(0) \in V, \gamma(1) \in W\}$.

Definition 1.20. Let γ be a piecewise smooth curve. The energy of γ is given by $\mathrm{E}(\gamma) = \int_0^1 |\dot{\gamma}|^2 dt$ and the length of γ is given by $\mathrm{L}(\gamma) = \int_0^1 |\dot{\gamma}| dt$.

Remark. This definition makes sense even though γ is only a piecewise smooth curve. △

Remark. The spaces $P(M)$ and $P(M, V, W)$ are topologized as follows. Let d_M be the metric associated to the Riemannian metric on M. Let γ_1, γ_2 be curves in $P(M)$ or $P(M, V, W)$ and let s_1 be the arc length function of γ_1 and let s_2 be the arc length function of γ_2. Then we give $P(M)$ and $P(M, V, W)$ the metric

$$\mathrm{d}(\gamma_1, \gamma_2) = \max_{0 \le t \le 1} \mathrm{d}_M(\gamma_1, \gamma_2) + \sqrt{\int_0^1 \Big(\frac{ds_1}{dt} - \frac{ds_2}{dt}\Big)^2 dt}\,.$$

The required topology is the topology induced by the above metric. E is continuous when $P(M)$ and $P(M, V, W)$ are given this metric. △

Definition 1.21. Formally, the tangent space of $P(M)$ and $P(M, V, W)$ at a point γ are

$$T_\gamma P(M) = \{Y \mid Y \text{ piecewise smooth vector field along } \gamma\}$$

and

$$T_\gamma P(M, V, W) =$$
$$\{Y \mid Y \text{ piecewise smooth vector field along } \gamma,\ Y(0) \in T_{\gamma(0)}V,\ Y(1) \in T_{\gamma(1)}W\}$$

In [M] the author considers the path space $P(M, p, q)$ for fixed $p, q \in M$ and calculates the first and second variation of the energy function $\mathrm{E} : P(M, p, q) \longrightarrow \mathbb{R}$. In our cases the first and second variation of $\mathrm{E} : P(M, V, W) \longrightarrow \mathbb{R}$ and $\mathrm{E} : P(M) \longrightarrow \mathbb{R}$ is essentially calculated as in [M], the only differences is that one has to keep track of the contributions that come from varying the endpoints.

Lemma 1.22. *Let Y, Y_1, Y_2 be tangent vectors at γ in either $P(M)$ or $P(M, V, W)$. The first variation of* E *is given by*

$$\frac{1}{2}\,\mathrm{E}_*(Y) = \langle Y, \dot{\gamma}\rangle|_0^1 - \sum_{t_i} \langle Y(t_i), \Delta_{t_i}\dot{\gamma}\rangle - \int_0^1 \Big\langle Y, \frac{D\dot{\gamma}}{dt}\Big\rangle dt,$$

where $\Delta_{t_i}\dot{\gamma} = \lim_{t \longrightarrow t_i^+} \dot{\gamma}(t) - \lim_{t \longrightarrow t_i^-} \dot{\gamma}(t)$ and the sum is over all break points of γ.

The second variation of E *at a critical point γ is given by*

$$\frac{1}{2}\,\mathrm{E}_{**}(Y_1, Y_2) =$$
$$-\sum_{t_i} \Big\langle Y_2(t_i), \Delta_{t_i}\frac{DY_1}{dt}\Big\rangle - \int_0^1 \Big\langle Y_2, \frac{D^2Y_1}{dt^2} + R(\dot{\gamma}, Y_1)\dot{\gamma}\Big\rangle dt,$$

where R is the curvature tensor of M. Similar to above $\Delta_{t_i}(DY_1/dt) = \lim_{t \longrightarrow t_i^+}(DY_1/dt) - \lim_{t \longrightarrow t_i^-}(DY_1/dt)$ and the sum is over all break points of γ.

Remark. Now it is easy to see what the critical points of E in $P(M,V,W)$ are, namely the smooth geodesics that start and end perpendicular to V and W. △

Notation. We let

i) $P_c(M) = \mathrm{E}^{-1}([0,c]) \subseteq P(M)$

ii) $P_{<c}(M) = \mathrm{E}^{-1}([0,c)) \subseteq P(M)$.

iii) $P_c(M,V,W) = \mathrm{E}^{-1}([0,c]) \subseteq P(M,V,W)$

iv) $P_{<c}(M,V,W) = \mathrm{E}^{-1}([0,c)) \subseteq P(M,V,W)$.

We will construct finite dimensional manifolds that approximates the spaces $P_{<c}(M)$ and $P_{<c}(M,V,W)$ up to homotopy equivalence. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be a partition of $[0,1]$. Let $P(M)(t_0,\dots,t_k)$ and $P(M,V,W)(t_0,\dots,t_k)$ be the set of curves

i) $P(M)(t_0,\dots,t_k) = \{\gamma \in P(M) \mid \gamma_{|[t_{i-1},t_i]} \text{ minimal geodesic for all } i = 1,\dots,k\}$

ii) $P(M,V,W)(t_0,\dots,t_k) =$
$\{\gamma \in P(M,V,W) \mid \gamma_{|[t_{i-1},t_i]} \text{ minimal geodesic for all } i = 1,\dots,k\}$

We shall need the following two theorems.

Theorem 1.23. *Let M, V and W be as above.*

(1) Let c be a fixed positive number such that $P_c(M,V,W) \neq \emptyset$. Then for all sufficiently fine subdivisions of $[0,1]$ the set

$$P_{<c}(M,V,W)(t_0,\dots,t_k) = P(M,V,W)(t_0,\dots,t_k) \cap P_{<c}(M,V,W)$$

can be given the structure of a smooth finite dimensional manifold.

(2) Let M be compact and let c be a fixed positive number such that $P_c(M) \neq \emptyset$. Then for all sufficiently fine subdivisions of $[0,1]$ the set

$$P_{<c}(M)(t_0,\dots,t_k) = P(M)(t_0,\dots,t_k) \cap P_{<c}(M),$$

can be given the structure of a smooth finite dimensional manifold.

Proof: The proof is essentially given in [M, Lemma 16.1]. We sketch the proof of (2).

For $0 \le a < b \le 1$ we let $\mathrm{L}_a^b(\omega)$ and $\mathrm{E}_a^b(\omega)$ denote the length and energy of ω from a to b. Let

$$\mathrm{inj}(p) = \sup\{\rho > 0 \mid \exp_p \text{ is defined on } B_\rho(0) \subseteq T_pM \text{ and injective}\}$$

be the injectivity radius of $p \in M$ and let $\operatorname{inj}(M) = \inf_{p \in M} \operatorname{inj}(p)$ be the injectivity radius of M. Since M is compact [C, Theorem 1.3] implies that the injectivity radius of M is strictly positive.

Choose a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ such that $t_i - t_{i-1} < (\operatorname{inj}(M)^2)/c$. Hence, for each broken geodesic $\omega \in P_{<c}(M)(t_0, \ldots, t_k)$ we have

$$(\mathrm{L}_{t_{i-1}}^{t_i}(\omega))^2 \leq (t_i - t_{i-1})\mathrm{E}_{t_{i-1}}^{t_i}(\omega) \leq (t_i - t_{i-1})\mathrm{E}(\omega) \leq (t_i - t_{i-1})c < \operatorname{inj}(M)^2.$$

Thus the geodesic $\omega_{|[t_{i-1}t_i]}$ is uniquely and differentiably determined by the two endpoints. Hence ω is uniquely determined by the $(k+1)$-tuple $(\omega(t_0), \omega(t_1) \ldots, \omega(t_k)) \in M \times M \times \cdots \times M$. The map $\omega \longmapsto (\omega(t_0), \omega(t_1) \ldots, \omega(t_k))$ defines a homeomorphism between $P_{<c}(M)(t_0, \ldots, t_k)$ and some open subset of the product $M \times M \times \cdots \times M$. We endow $P_{<c}(M)(t_0, \ldots, t_k)$ with a differentiable structure by requiring that the correspondence $\omega \longmapsto (\omega(t_0), \omega(t_1) \ldots, \omega(t_k))$ is a diffeomorphism. □

Notation. Denote $P_{<c}(M)(t_0, \ldots, t_k)$ by $B_c^k(M)$ and denote $P_{<c}(M, V, W)(t_0, \ldots, t_k)$ by $B_c^k(M, V, W)$.

The following theorem is basic in Morse Theory.

Theorem 1.24.

(1) The map $\mathrm{E}' = \mathrm{E}_{|B_c^k(M,V,W)} : B_c^k(M, V, W) \longrightarrow [0, c)$ is smooth and for each $a < c$ the set $\mathrm{E}'^{-1}([0, a]) = B_a(M, V, W)$ is compact (E' is proper). $B_a(M, V, W)$ is a deformation retract of the corresponding set $P_a(M, V, W)$ and $B_c^k(M, V, W)$ is a deformation retract of $P_{<c}(M, V, W)$. Furthermore the critical points of E' are the same as the critical points of E in $B_c^k(M, V, W)$. The index of the Hessian of E' at a critical point γ is the same as the index of E_{**} at γ.

(2) The map $\mathrm{E}' = \mathrm{E}_{|B_c^k(M)} : B_c^k(M) \longrightarrow [0, c)$ is smooth and for each $a < c$ the set $\mathrm{E}'^{-1}([0, a]) = B_a(M)$ is compact (E' is proper). $B_a(M)$ is a deformation retract of the corresponding set $P_a(M)$ and $B_c^k(M)$ is a deformation retract of $P_{<c}(M)$. Furthermore the critical points of E' are the same as the critical points of E in $B_c^k(M)$. The index of the Hessian of E' at a critical point γ is the same as the index of E_{**} at γ.

Proof: The proof is essentially given in [M, Chapter 14, 16]. We will not give any further details. □

The following proposition is the essential tool in the proofs of the connectedness theorems.

Proposition 1.25. *Let M be a smooth manifold a let $f\colon M \longrightarrow \mathbb{R}$ be a smooth map with minimum 0. Assume that $M_c = f^{-1}([0,c])$ is compact for all c. If the set of minimal points M_0 has a neighbourhood U with a retraction $r\colon U \longrightarrow M_0$ and every critical point in $M \setminus M_0$ has index $\geq \lambda$ then*[2]

$$\pi_i(M, M_0) = 0 \quad \text{for} \quad i < \lambda.$$

Proof: The proof in [M, Lemma 22.5] also applies to this situation. □

Remark. In particular, if M_0 is a submanifold, M_0 is a retract of some open tubular neighbourhood U in M.

It is important to notice that the energy function $\mathrm{E}' : B_c^k(M,V,W) \longrightarrow [0,c)$ satisfies the conditions of the proposition if, for example, $\mathrm{E}^{-1}(0) = V \cap W$ is a submanifold. The intersection $V \cap W$ is a submanifold if $V \cap W$ is tranverse or if both V and W are totally geodesic. △

[2] There does not seem to be a completely satisfactory way of defining $\pi_0(X,Y)$. The sets $\pi_1(X,Y)$ and $\pi_0(X)$ are not generally groups. By the expressions $\pi_1(X,Y) = 0$ and $\pi_0(X) = 0$ we mean that the sets $\pi_1(X,Y)$ and $\pi_0(X)$ consist of only one point, namely the basepoint. Exactness still makes sense when the sets are not groups: The image is equal to the inverse image of the basepoint.

Chapter 2

Positive Curvature and the Connectedness Principle

We will use the results of the previous chapter to prove the classical theorems of Frankel and Synge and a recent and important connectedness theorem by Wilking, see [W I]. Wilking's theorem simply states that the inclusion of a compact totally geodesic embedded codimension k submanifold in a positively curved m-manifold is $(m-2k+1)$-connected. We will also see that Frankel's theorem is a simple corollary of Wilking's theorem.

2.1 Theorems of Frankel and Synge

In this section we prove the theorems of Frankel and Synge. We also prove a generalization of Synge's theorem due to Weinstein. We start by stating and proving Frankel's theorem, see [F].

Theorem 2.1. *Assume that M^m has positive sectional curvature. Let V^s and W^r be two compact totally geodesic submanifolds. If $r+s \geq m$ then $V \cap W \neq \emptyset$.*

Proof: Assume on the contrary that V and W do not intersect. Then since M is complete and V, W are compact there exists a shortest geodesic γ joining V and W of length $l > 0$. Since γ is a geodesic of shortest length, γ is a critical point of E. It follows from the first variation formula that $\dot\gamma(0) \perp T_{\gamma(0)}V$ and $\dot\gamma(1) \perp T_{\gamma(1)}W$. We will construct a variation of γ with variation vector field X such that $\mathrm{E}_{**}(X,X) < 0$. This would give the desired contradiction since γ is minimizing.

Let $p = \gamma(0)$ and $q = \gamma(1)$. Let $P\colon T_pM \longrightarrow T_qM$ be parallel translation along γ. By parallel translating T_pV along γ we get a subspace $P(T_pV) \subseteq T_qM$. Since P is an isometry and $T_pV \perp \dot\gamma(0)$ we know that $P(T_pV) \perp \dot\gamma(1)$. The tangent space T_qW is also perpendicular to $\dot\gamma(1)$, hence the intersection $P(T_pV) \cap T_qW$ has dimension

$$\dim(P(T_pV) \cap T_qW) \geq r+s-(m-1) \geq 1.$$

This implies that there is a unit vector $X_0 \in T_pV$ whose parallel translate is a unit vector in T_qW. Denote the vector field $P(X_0)$ along γ by X.

Using the second variation formula, we see that $\mathrm{E}_{**}(X,X) < 0$, since M is positively curved and X is parallel. We finish the proof by constructing a variation of γ with the endpoints in V and W and with variation vector field X. Define a variation $\alpha(s,t) = \exp_{\gamma(t)}(sX)$. Clearly α has X as variation vector field. Since V and W are totally geodesic we see that the endpoints $\alpha(s,0) = \exp_{\gamma(0)}(sX)$ and $\alpha(s,1) = \exp_{\gamma(1)}(sX)$ are in V respectively in W. □

We now state and prove Synge's theorem, see [S]. The argument in the proof is classical, and is often called a "Synge type argument".

Theorem 2.2. *Assume that M has positive sectional curvature.*

1. *If M is even dimensional and orientable, then M is simply connected.*
2. *If M is odd dimensional, then M is orientable.*

Proof: Synge's original proof used the length functional, we will use the energy functional. (1) We first note that it is enough to consider free homotopy classes. Assume on the contrary that M is not simply connected. Since M is not simply connected there exists some nontrivial free homotopy class $[h]$. We need the following claim.

Claim 1. The free homotopy class $[h]$ contains elements which are piecewise smooth. The number

$$\lambda = \inf\{\mathrm{L}(\sigma) \mid \sigma \in [h],\ \sigma \text{ piecewise smooth}\}$$

is strictly positive. Furthermore, there exists an element with length equal to λ and this element is a smooth geodesic.

Proof of Claim (1): The Bonnet and Myers' theorem implies that M is compact since the sectional curvature is positive, so the claim follows from [C, Theorem 4.12]. △

Denote that minimal geodesic in $[h]$ by γ. By parallel translating along γ we obtain an isometry $P\colon T_{\gamma(0)}M \longrightarrow T_{\gamma(1)}M = T_{\gamma(0)}M$. The vector field $\dot\gamma$ is parallel along γ and since P is an isometry it preserves the orthogonal complement to $\dot\gamma$. Let $N = \dot\gamma(0)^\perp \subseteq T_{\gamma(0)}M$ and note that $N = \dot\gamma(1)^\perp \subseteq T_{\gamma(1)}M$ since γ is smooth and closed. By restricting P to N we get an orientation preserving isometry $P_|\colon N \longrightarrow N$. Since M is even dimensional, N is odd dimensional. Identifying N with some $\mathbb{R}^{2k-1}$, we see that $P_|$ is an orientation preserving orthogonal transformation.

The eigenvalues of $P_|$ are the roots of the characteristic polynomial and hence if z is an eigenvalue $\bar z$ is also an eigenvalue. Also, the eigenvalues all have norm 1. An orientation preserving orthogonal transformation has determinant 1. Hence we see that an orientation preserving orthogonal transformation on an odd dimensional vector space has eigenvalue 1. Since N is odd dimensional it follows that $P_|$ has eigenvalue 1, that is, there exists a v in N such that $P_|(v) = v$.

By parallel translating v along γ we get a parallel vector field V along γ. Consider the variation of γ given by $\alpha(t,s) = \exp_{\gamma(t)}(sV)$ with $\alpha(0,s) = \alpha(1,s) = \gamma(0)$. Using the second variation formula for E, [M, Theorem 13.1], we see that $\mathrm{E}_{**}(V,V) < 0$ since V is parallel and M is positively curved. Hence for sufficiently small $s > 0$, $\mathrm{E}(\alpha_s) < \mathrm{E}(\gamma)$. This implies that there are nearby curves in the same homotopy class as γ (α_s is the desired homotopy) which are shorter. This is a contradiction to the minimality of γ.
(2) The proof is similar to the proof of (1). Assume that M is not orientable then we will obtain a contradiction. As M is complete, M consists of one component which is not orientable. Since M is not orientable, Proposition 1.8 implies that M is not simply connected.

Choose as in (1) a geodesic γ of minimal length in some nontrivial free homotopy class. Consider parallel transport along γ and construct a subspace $N = \dot\gamma(0)^\perp = \dot\gamma(1)^\perp \subseteq T_{\gamma(0)}M = T_{\gamma(1)}M$. Since M is not orientable, parallel transport reverses orientation. The map $P_| \colon N \longrightarrow N$ is an orthogonal transformation and since the dimension of M is odd, the dimension of N is even. Since $P_|$ reverses orientation and the dimension of N is even, $P_|$ must have eigenvalue 1 and hence a fixed point. The proof is now finished as in (1).□

The following theorem due to Weinstein is a generalization of Synge's theorem.

Theorem 2.3. *Let M be an oriented Riemannian manifold with positive sectional curvature and let $f \colon M \longrightarrow M$ be a fixed point free isometry.*

1. *If M is even dimensional, then f reverses orientation.*

2. *If M is odd dimensional, then f preserves orientation.*

Proof: We will only prove (1) as (2) is proved similarly.
(1) Assume on the contrary that f preserves orientation. Pick $p \in M$ and let $q = f(p) \neq p$. Since M is complete we choose a minimal geodesic γ from p to q and extend γ smoothly past both p and q. The map f induces an isometry $f_* \colon T_pM \longrightarrow T_qM$. Consider the complement $\dot\gamma(0)^\perp \subseteq T_pM$ and $\dot\gamma(1)^\perp \subseteq T_qM$. By identifying T_pM and T_qM with $\mathbb{R}^{2k}$ we see that $f_* \colon \mathbb{R}^{2k} \longrightarrow \mathbb{R}^{2k}$ is an orthogonal transformation.

Since f_* is an isometry it preserves this complement, hence inducing an isometry $f_{|*} \colon \mathbb{R}^{2k-1} \longrightarrow \mathbb{R}^{2k-1}$ which again is an orthogonal transformation. Since $f_{|*}$ preserves orientation and $\mathbb{R}^{2k-1}$ is odd dimensional, it has a fixed point, v.

Extend v to a parallel vector field V along γ. Consider the variation of γ with variation vector field V given by $\alpha(s,t) = \exp_{\gamma(t)}(sV)$ (keeping endpoints fixed). As V is parallel and M has positive sectional curvature the second variation formula implies that $\mathrm{E}_{**}(V,V) <$

0. Hence for small values of s the curves α_s satisfies $\mathrm{E}(\alpha_s) < \mathrm{E}(\gamma)$. This is a contradiction, since γ is minimal, hence f reverses orientation. □

2.2 Wilking's Connectedness Theorem

In this section we prove Wilking's connectedness theorem and show that the connectedness theorem has consequences for the cohomology ring of the manifold.

Theorem 2.4. *Assume that M^m has positive sectional curvature and that the submanifolds are path connected.*

1. *Suppose $N^{m-k} \subset M^m$ is a compact totally geodesic embedded submanifold. Then the inclusion map $i\colon N^{m-k} \longrightarrow M^m$ is $(m-2k+1)$-connected.*

2. *Suppose $N_1^{m-k_1}, N_2^{m-k_2} \subset M^m$ are two compact totally geodesic embedded submanifolds with $k_1 \leq k_2$ and $k_1 + k_2 \leq m$. Then the intersection is a compact totally geodesic embedded submanifold as well and the inclusion $i\colon N_1^{m-k_1} \cap N_2^{m-k_2} \longrightarrow N_2^{m-k_2}$ is $(m-k_1-k_2)$-connected.*

Remark. Since M has positive sectional curvature, M is compact by the Bonnet and Myers Theorem. △

Remark. An example where the conclusion in the theorem is optimal is the following. According to [W I] the Wallach flag $M^{24} = \mathsf{F}_4/\mathsf{Spin}(8)$ contains a 15-dimensional totally geodesic submanifold N^{15} diffeomorphic to S^{15}. The manifold M^{24} is 7-connected but not 8-connected. Hence the inclusion map is also 7-connected but not 8-connected. △

Remark. Let $\Omega(M)$ be the loop space of M. We ought to pick a basepoint and write $\Omega(M,*)$, but since M is path connected, $\Omega(M,p_1)$ is homotopy equivalent to $\Omega(M,p_2)$ for all $p_1, p_2 \in M$.

We will use the fact that $\pi_{i+1}(M) = \pi_i(\Omega(M))$ many times throughout the text. To see this, let $P(M,*)$ be the set of curves in M ending at $*$. By [B, Corollary 6.17] the map $ev_0\colon P(M,*) \longrightarrow M$ given by $\gamma \longmapsto \gamma(0)$ is a fibration. The space $P(M,*)$ is contractible since a contraction is given by truncating the curves to their endpoint $*$. Since $P(M,*)$ is contractible $\pi_i(P(M,*)) = 0$, for all i. By considering the long exact sequence for the fibration $\Omega(M) \longrightarrow P(M,*) \longrightarrow M$

$$\longrightarrow 0 \longrightarrow \pi_{i+1}(M) \longrightarrow \pi_i(\Omega(M)) \longrightarrow 0 \longrightarrow$$

the claim now follows. △

Proof (Theorem 2.4): Notice first that, since M is path connected, (1) contains no new information, if $m - 2k + 1 < 1$, hence we assume that $m - 2k + 1 \geq 1$. We also note that, since $m \geq k_1 + k_2$, the intersection $N_1 \cap N_2$ is non-empty by Frankel's theorem, Theorem 2.1.
(1) We will use the set-up described in Section 1.2 with $V = N$ and $W = N$. There is a natural embedding of N in $P(M, N, N)$ as the set of constant paths in N. This set is also identified with $\mathrm{E}^{-1}(0)$. We need the following claim.

Claim 2. The inclusion $N \longrightarrow P(M, N, N)$ is $(m - 2k)$-connected.

Proof of Claim (2): As noted in Section 1.2, the critical points of E in $P(M, N, N)$ are the geodesics starting and ending orthogonally to N.

Assume that $\gamma \in P(M, N, N)$ is a non-trivial critical point, i.e. not a constant path. We will estimate the index of such a non-trivial critical point. Let $W \in T_\gamma P(M, N, N)$ be a smooth parallel vector field along γ. Then we have

$$\frac{1}{2}\,\mathrm{E}_{**}(W, W) = -\int_0^1 \langle W, R(\dot\gamma, W)\dot\gamma\rangle dt.$$

Since the sectional curvature is assumed to be positive, $\mathrm{E}_{**}(W, W) < 0$ for such a W.

A parallel vector field along a critical point $\gamma \in P(M, N, N)$ which satisfies $W(0) \in T_{\gamma(0)}N$ and $W(1) \in T_{\gamma(1)}N$ is orthogonal to $\dot\gamma(t)$ for all $t \in [0, 1]$. Using this fact, we see that we can estimate the index of a non-trivial critical point by estimating the number of linearly independent smooth parallel vector fields along γ and orthogonal to $\dot\gamma$. Since $W(0) \in T_{\gamma(0)}N$ and $W(1) \in T_{\gamma(1)}N$ and the codimension of N is k, the number of linearly independent smooth parallel vector fields along γ and orthogonal to $\dot\gamma$ is $m - 1 - (k - 1) - (k - 1) = m - 2k + 1$.

We must construct a variation of γ with W as variation vector field and with the endpoints in N. Such a variation with W as variation vector field is given by $\alpha(t, s) = \exp_{\gamma(t)}(sW)$. Since N is totally geodesic the curves $\alpha(0, s)$ and $\alpha(1, s)$ are geodesics in N which means the endpoints of the variation are in N, as desired. Hence the index of a non-trivial critical point is at least $m - 2k + 1$.

From Theorem 1.24 we know that E' is a smooth and proper map and since the set $\mathrm{E}'^{-1}(0) = N$ is a manifold Proposition 1.25 implies that for every $c \geq 0$ the relative homotopy groups $\pi_i(B^k_c(M, N, N), N) = 0$, for $i \leq m - 2k$, since the index of the critical points of E are the same as the index of the critical points of E' which are $\geq m - 2k + 1$. This means that up to homotopy $B^k_c(M, N, N)$ is obtained from N by attaching cells of dimension at least $m - 2k + 1$. From Theorem 1.24 we also know that there is a homotopy equivalence between $B^k_c(M, N, N)$ and $P_{<c}(M, N, N)$. This means that for all c with $P_c(M, N, N) \neq \emptyset$, $P_{<c}(M, N, N)$ is homotopy equivalent to a relative CW-complex $N \cup e^1 \cup e^2 \cup \cdots$ where the dimension of e^i is at least $m - 2k + 1$. Of course, for c small

enough it is possible that there are no critical values between 0 and c. In that case we do not attach any cells.

Let $c_0 < c_1 < c_2 < \cdots$ be a sequence of positive real numbers with $\lim_{i\longrightarrow\infty} c_i = \infty$. Consider the directed system $\{P_{<c_i}(M,N,N)\}$ with union $P(M,N,N)$. The space $P(M,N,N)$ is paracompact since it is a metric space, see [MS, Theorem of A. H. Stone], and every point of $P(M,N,N)$ lies in the interior of some $P_{<c_i}(M,N,N)$. Hence [M, Example A1] implies that $P(M,N,N)$ is the homotopy direct limit of $\{P_{<c_i}(M,N,N)\}$. Since $P_{<c_i}(M,N,N)$ is homotopy equivalent to a relative CW-complex $K_i = N \cup e^{i,0} \cup e^{i,1} \cup \cdots$ where the dimension of $e^{i,j}$ is at least $m-2k+1$ we have an increasing sequence of relative CW-complexes $K_0 \subset K_1 \subset K_2 \subset \cdots$. Let K be the union of the K_is. [M, Example A2] implies that K is the homotopy direct limit of the directed system $\{K_i\}$. As all the CW-complexes K_i are obtained from N by attaching cells of dimension at least $m-2k+1$, K is also obtained from N by attaching cells of dimension at least $m-2k+1$. For all i, the vertical maps in the diagram

$$\begin{array}{ccccccccc}
\longrightarrow & P_{c_i}(M,N,N) & \longrightarrow & P_{c_{i+1}}(M,N,N) & \longrightarrow & \cdots & \longrightarrow & P(M,N,N) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & K_i & \longrightarrow & K_{i+1} & \longrightarrow & \cdots & \longrightarrow & K
\end{array}$$

are homotopy equivalences. Theorem 1.14 now implies that the map $P(M,N,N) \longrightarrow K$ is a homotopy equivalence. This means that the relative homotopy groups

$$\pi_i(P(M,N,N),N) = 0 \quad \text{for} \quad i \leq m-2k.$$

Since N is path connected it follows from the fact that $\pi_1(P(M,N,N),N) = 0$ and the long exact sequence

$$\to \pi_{i+1}(P(M,N,N),N) \to \pi_i(N) \to \pi_i(P(M,N,N)) \to \pi_i(P(M,N,N),N) \to$$

for the pair $(P(M,N,N),N)$ that the inclusion induces a bijection

$$i_*\colon \pi_0(N) \longrightarrow \pi_0(P(M,N,N)).$$

By considering the same long exact sequence it follows that the inclusion $i_*\colon \pi_i(N) \longrightarrow \pi_i(P(M,N,N))$ is $(m-2k)$-connected, since

$$\pi_i(P(M,N,N),N) = 0 \quad \text{for } i \leq m-2k.$$

This ends the proof of the claim. △

Claim 3. There is an isomorphism $\pi_i(P(M,N,N),N) \cong \pi_{i+1}(M,N)$.

Assuming this claim we can now finish the proof of (1). The inclusion induces a bijection $i_*\colon \pi_0(N) \longrightarrow \pi_0(M)$, since both N and M are path connected. Since $\pi_i(P(M,N,N),N) = 0$, for $i \leq m-2k$ it follows from Claim 3 that $\pi_i(M,N) = 0$, for $i \leq m-2k+1$. The long exact homotopy sequence for the pair (M,N)

$$\longrightarrow \pi_{i+1}(M,N) \longrightarrow \pi_i(N) \longrightarrow \pi_i(M) \longrightarrow \pi_i(M,N) \longrightarrow$$

implies that the map $i_*\colon \pi_i(N) \longrightarrow \pi_i(M)$ is an isomorphism for $0 < i < m-2k+1$ and induces a surjection on the $(m-2k+1)$th homotopy group. This finishes the proof of (1).

Proof of Claim (3): There is a bijection between the set of maps $(D^i, S^{i-1}) \longrightarrow (M,N)$ and the set of maps $(D^{i-1}, S^{i-2}) \longrightarrow (P(M,N,N), P(N))$. This implies that there is an isomorphism $\pi_i(M,N) \cong \pi_{i-1}(P(M,N,N),P(N))$. Since $P(N)$ is canonically homotopy equivalent to N by truncating the paths to their start point the two long exact homotopy sequences for the pairs $(P(M,N,N),P(N))$ and $(P(M,N,N),N)$ and the Five-Lemma implies that $\pi_i(P(M,N,N),P(N)) \cong \pi_i(P(M,N,N),N)$. Combining the two isomorphisms we get the desired result. △

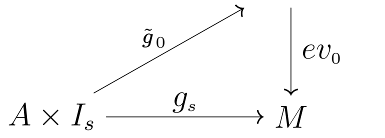
(2) Notice first that it is clear that $N_1 \cap N_2$ is a compact totally geodesic submanifold. By considering the space $P(M,N_1,N_2)$ and noticing that $N_1 \cap N_2 = \mathrm{E}^{-1}(0)$ an argument similar to the argument in the proof of Claim 2 above yields that the inclusion $i\colon N_1 \cap N_2 \longrightarrow P(M,N_1,N_2)$ is $(m-k_1-k_2)$-connected.

Consider now the space $P(M,M,N_2)$ of curves starting in M and ending in N_2. To finish the proof of (2) we need the following claim.

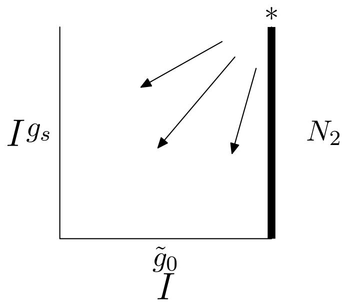
Claim 4. The map $i\colon P(M,N_1,N_2) \longrightarrow P(M,M,N_2)$ is $(m-2k_1+1)$-connected.

We can now finish the proof of (2). The map $i\colon N_1 \cap N_2 \longrightarrow P(M,N_1,N_2)$ is $(m-k_1-k_2)$-connected and by Claim 4 the map $i\colon P(M,N_1,N_2) \longrightarrow P(M,M,N_2)$ is $(m-2k_1+1)$-connected. Hence the composition $N_1 \cap N_2 \longrightarrow P(M,M,N_2)$ is $(m-k_1-k_2)$-connected as $k_2 \geq k_1$. The space $P(M,M,N_2)$ is homotopy equivalent to N_2 by truncating the paths to their endpoint, hence we see that $i\colon N_1 \cap N_2 \longrightarrow N_2$ is $(m-k_1-k_2)$-connected, which finishes the proof of (2).

Proof of Claim (4): We will see that the fact that $i\colon N_1 \longrightarrow M$ is $(m-2k_1+1)$-connected implies the claim. We will first show that we have a fibration $ev_0\colon P(M,M,N_2) \longrightarrow M$ given by evaluating the curves at their start point. The fibre of this fibration is $P(M,p,N_2)$. Let s be the homotopy variable. Let A be a topological space and consider the following diagram



The curves end in N_2 thus $\tilde{g}_0(a,1) \in N_2$. The map $\tilde{g}_0$ lifts g_0 that is $g(a,0) = ev_0(\tilde{g}(a,t)) = \tilde{g}(a,0)$. We have to find a homotopy $\tilde{g}_s \colon A \times I_s \longrightarrow P(M,M,N_2)$ of $\tilde{g}_0$ that lifts g_s, that is, a map $\tilde{g} \colon A \times I \times I \longrightarrow M$ such that $ev_0(\tilde{g}) = \tilde{g}(a,0,s) = g(a,s)$ and $\tilde{g}(a,t,0) = \tilde{g}_0$. The curves end in N_2 hence $\tilde{g}(a,1,s) \in N_2$. Define the image of $\tilde{g}$ on $I \times I$ to be the inverse image of the projection from the point $*$ onto the three axis.



Notice that $*$ is *not* supposed to touch the bold line!

The inclusion $i \colon N_1 \longrightarrow M$ gives rise to a pull-back fibration

$$\begin{array}{ccccc} P(M,p_1,N_2) & \longrightarrow & P(M,N_1,N_2) & \xrightarrow{\pi} & N_1 \\ = \downarrow & & \tilde{i} \downarrow & & \downarrow i \\ P(M,p_1,N_2) & \longrightarrow & P(M,M,N_2) & \xrightarrow{ev_0} & M \end{array}$$

The spaces $P(M,M,N_2)$ and $P(M,N_1,N_2)$ are path connected, since M is path connected. Hence the inclusion induces a bijection $i_* \colon \pi_0(P(M,N_1,N_2)) \longrightarrow \pi_0(P(M,M,N_2))$. Since the map $i \colon N_1 \longrightarrow M$ is $(m-2k_1+1)$-connected and the fibres of the two fibrations are equal an easy application of the long exact homotopy sequence for the two fibrations together with the Five-Lemma implies that $i_* \colon \pi_i(P(M,N_1,N_2)) \longrightarrow \pi_i(P(M,M,N_2))$ is an isomorphism for $0 < i < m - 2k_1$ and a surjection for $i = m - 2k_1 + 1$. △

This ends the proof of the theorem. □

Remark. Let M and N be as in Theorem 2.4. Let G be a Lie group acting on M by isometries and assume that N is fixed pointwise by the action of G. Then the inclusion $i \colon N \longrightarrow M$ is $(m-2k+1+\delta(G))$-connected where $\delta(G)$ is the dimension of the principal orbit. The proof can be found in [W I, Theorem 1].

The fixed point set of an isometric group action is a totally geodesic submanifold. Theorem 2.4 turns out to be a very powerful tool to examine isometric group actions on manifolds. We will, however, not deal with group actions here. △

We have an easy corollary of Theorem 2.4.

Corollary 2.5. *Let the assumptions be as in Theorem 2.4. If M has positive lth Ricci curvature then the map in (1) is $(m-2k+2-l)$-connected and the map in (2) is $(m-k_1-k_2+1-l)$-connected.*

Proof: The only place in the proof where the curvature assumption is used is in the estimation of the indices. An argument similar to the one given in the proof of Proposition 3.11 gives the desired estimates on the indices. □

Remark. Frankel's theorem, Theorem 2.1, follows from Wilking's Theorem 2.4 since part (2) of that theorem implies that the inclusion $i\colon N_1 \cap N_2 \longrightarrow N_2$ is 0-connected since $k_1 + k_2 \leq m$, hence induces a surjection $\pi_0(N_1 \cap N_2) \longrightarrow \pi_0(N_2)$. Since $\pi_0(\emptyset) = \emptyset$ the theorem follows. △

Remark. In the proof of Synge's theorem, Theorem 2.2, one could also use an approach more similar to the proof of Theorem 2.4. If we consider the loop space of M then we see by the argument in the proof of Synge's theorem that all nontrivial critical points of E have index ≥ 1. Hence $\pi_0(\Omega(M)) = 0$ which implies that $\pi_1(M) = 0$. △

We will now see what consequences Theorem 2.4 has for the structure of the cohomology ring. This is formulated in the next theorem.

Theorem 2.6. *Let M^m be a compact oriented manifold and let N^{m-k} be an embedded compact oriented submanifold. Suppose the inclusion $i\colon N^{m-k} \longrightarrow M^m$ is $(m-k-l)$-connected and $m-k-2l > 0$. Let $[N] \in H_{m-k}(M;\mathbb{Z})$ be the image of the fundamental class of N in $H_*(M;\mathbb{Z})$ and let $e \in H^k(M;\mathbb{Z})$ be its Poincare dual. Then the homomorphism*

$$\cup e\colon H^i(M;\mathbb{Z}) \longrightarrow H^{i+k}(M;\mathbb{Z})$$

is surjective for $l \leq i < m-k-l$ and injective for $l < i \leq m-k-l$.

Remark. The pull-back of e to $H^k(N,\mathbb{Z})$ is the Euler class of the normal bundle $\nu(N)$ of N in M. It should not cause any confusion that we denote both the Euler class and the above class by e. △

Proof: First we show that the map

$$\cup e\colon H^i(M;\mathbb{Z}) \longrightarrow H^{i+k}(M;\mathbb{Z})$$

is given by the composition of the following four maps

$$\begin{array}{ccccccccc} H^i(M;\mathbb{Z}) & \xrightarrow{i^*} & H^i(N;\mathbb{Z}) & \xrightarrow{PD_N} & H_{n-k-i}(N;\mathbb{Z}) & \xrightarrow{i_*} & H_{n-k-i}(M;\mathbb{Z}) & \xrightarrow{PD_M^{-1}} & H^{i+k}(M;\mathbb{Z}) \\ & w \longmapsto i^*(w) & & x \longmapsto x \cap [N] & & y \longmapsto i_*(y) & & z \cap [M] \longmapsto z & \end{array}$$

Let $w \neq 0 \in H^i(M, \mathbb{Z})$. Using the formulas for $\cap$ and $\cup$ we get

$$PD_M^{-1}(i_*(PD_N(i^*(w)))) = PD_M^{-1}(i_*(i^*(w) \cap [N])) = PD_M^{-1}(w \cap i_*([N]))$$

and

$$w \cup e = w \cup (PD_M^{-1}(i_*([N]))) = PD_M^{-1}(w \cap i_*([N])).$$

By Theorem 1.11 the maps PD_N and PD_M^{-1} are isomorphisms for all i. By assumption the inclusion $i\colon N \longrightarrow M$ is $(m-k-l)$-connected. From the proof of [H, Theorem 4.21] this implies that $H_i(M, N; \mathbb{Z}) = H^i(M, N; \mathbb{Z}) = 0$ for $i \leq m-k-l$. Hence by using the long exact homology sequence for the pair (M, N) we see that the map (1) is an isomorphism for $i < m-k-l$ and injective for $i = m-k-l$. By a similar argument the map (3) is an isomorphism for $i > l$ and surjective for $i = l$. □

Remark. We will also need the theorem with the coefficient ring replaced by an arbitrary unital ring R. The same proof applies, the only difference is that we must require M and N to be R-oriented. △

Chapter 3

The Twisted Path Space of a Manifold

In the previous chapter we proved the classical theorems of Frankel and Synge and indicated how these theorems follow from the connectedness principle. We also proved a recent theorem by Wilking and investigated its consequences on the cohomology ring of the manifold.

In this chapter we will apply Morse Theory to the twisted path space and in that way derive some new connectedness theorems, developed by Fang, Mendonca and Rong, see [FMR]. We will formulate connectedness theorems in terms of the so-called asymptotic index (to be defined below) of an isometric immersion. We will prove generalizations of Frankel and Wilking's theorems and show that under certain conditions an isometric immersion must be an embedding.

In this chapter, we assume that M and N, N_1, N_2 are compact and path connected Riemannian manifolds.

3.1 Construction of The Twisted Path Space

The following construction works for f an arbitrary continuous map, but the main application is where $f\colon N \longrightarrow M \times M$ an isometric immersion.

We start with the following basic lemma. As before, we let $P(M)$ be the path space of M.

Lemma 3.1. *Let $p\colon P(M) \longrightarrow M \times M$ be the projection given by $p(\gamma) = (\gamma(0), \gamma(1))$. Then the map $p\colon P(M) \longrightarrow M \times M$ is a (Hurewicz) fibration with homotopy fibre the loop space $\Omega(M)$ of M.*

Proof: The fibre $p^{-1}(p, q)$ of p is the set of curves from p to q and is homotopy equivalent to the loop space $\Omega(M)$ based at p by moving the endpoints of the curves along a curve from p to q. This is possible since M is path connected.

Since both $\{0,1\}$ and $[0,1]$ are CW-complexes the inclusion $i\colon \{0,1\} \longrightarrow [0,1]$ is a cofibration. It follows from [B, 6.13] that the induced map

$$\mathrm{Map}([0,1], M) \longrightarrow \mathrm{Map}(\{0,1\}, M) = M \times M$$

is a fibration. As $\mathrm{Map}([0,1], M)$ is homotopy equivalent to $P(M)$ the result follows. □

We can now define the twisted path space $P(M,f)$ of M with respect to f.

Definition 3.2. The twisted path space $P(M,f)$ is the pull-back of the fibration p by the map $f\colon N \longrightarrow M \times M$. That is,

$$P(M,f) = \{(x,\gamma) \mid x \in N, \gamma \in P(M); f(x) = (\gamma(0), \gamma(1))\}.$$

The space $P(M,f)$ is given the induced topology from the product $N \times P(M)$. Since the fibres of the fibration and the pull-back fibration are always equal we have a commutative diagram of fibrations

$$\begin{array}{ccccc} \Omega(M) & \longrightarrow & P(M,f) & \xrightarrow{\pi} & N \\ {\scriptstyle =}\downarrow & & {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \Omega(M) & \longrightarrow & P(M) & \xrightarrow{p} & M \times M \end{array}$$

Later we will need the following lemma.

Lemma 3.3. *Let $f\colon N \longrightarrow M \times M$ be a smooth map. Then $f^*B_c^k(M)$ is a smooth submanifold of $N \times B_c^k(M)$.*

Proof: Since M is compact and complete it follows from Theorem 1.23 that $B_c^k(M)$ is a smooth manifold. Formally the tangent space of $P(M)$ at a point γ is the set

$$T_\gamma P(M) = \{W \mid W \text{ piecewise smooth vector fields along } \gamma\}.$$

The differential of the map p at a point γ is $p_*(W) = (W(0), W(1))$. It is clear that that p_* is surjective. Hence $p\colon B_c^k(M) \longrightarrow M \times M$ is a submersion. Furthermore we have $f^*B_c^k(M) = (f \times p)^{-1}(\Delta)$, where Δ is the diagonal submanifold in $(M \times M) \times (M \times M)$. Since p is a submersion $f \times p$ is transversal to Δ. It is then a fact see [G I, Example 16.2] which we do not prove that $f^*B_c^k(M)$ is a smooth submanifold of $N \times B_c^k(M)$. One should add that the reference only deals with vector bundles, but the set-theoretical definition of the pull-back for vector bundles is the same as the one for fibrations, so the result applies here too. □

Notation. We will denote $f^*B_c^k(M)$ by $B_c^k(M,f)$ and $f^*P_{<c}(M)$ by $P_{<c}(M,f)$ etc.

3.2 Morse Theory

Let $f\colon N \longrightarrow M \times M$ be an isometric immersion. We start with the following basic definition.

Definition 3.4. Let $(x,\gamma) \in P(M,f)$. The energy of (x,γ) is $\mathrm{E}(x,\gamma) = \int_0^1 |\dot{\gamma}|^2 dt$

Definition 3.5. Formally, the tangent space of $P(M,f)$ at a point (x,γ) is the set

$$T_{(x,\gamma)}P(M,f) = \{(v,W) \mid v \in T_xN, W \in T_\gamma P(M);\ f_*(v) = (W(0),W(1))\}.$$

The last condition comes from the fact that $f(x) = (\gamma(0),\gamma(1))$ in the definition of the pull-back. The tangent space $T_{(x,\gamma)}P(M,f)$ can be identified with the set $\{(W(0),W(1))\}$ since f is an immersion.

Remark. Since the pull-back $P(M,f)$ is a subset of $N \times P(M)$ and the energy function only depends on the $P(M)$ factor the first and second variation formulas are the same as the ones stated in Section 1.2. △

The critical points of E are easy to calculate. We formulate that in the following lemma.

Lemma 3.6. *The (non-trivial) critical points of* E *in* $P(M,f)$ *are the points* (x,γ) *where* γ *is a smooth geodesic with* $(\dot{\gamma}(0),-\dot{\gamma}(1)) \perp f_*(T_xN)$.

Proof: Let $W \in T_{(x,\gamma)}P(M,f)$. The first variation formula is

$$\frac{1}{2}\,E_*(W) = \langle W,\dot{\gamma}\rangle|_0^1 - \sum_{t_i}\langle W(t),\Delta_{t_i}\dot{\gamma}(t_i)\rangle - \int_0^1 \Big\langle W,\frac{D\dot{\gamma}}{dt}\Big\rangle dt,$$

where the sum is over all break points of γ and $\Delta_{t_i}\dot{\gamma} = \lim_{t\longrightarrow t_i^+}\dot{\gamma}(t) - \lim_{t\longrightarrow t_i^-}\dot{\gamma}(t)$. For this to be zero for all W it is clearly necessary that γ is a smooth geodesic.

We consider the geodesic γ as the geodesic $(\gamma(t),\gamma(1-t))$ in the product $M\times M$. Under this identification the term $\langle W,\dot{\gamma}\rangle|_0^1$ becomes $\langle (W(0),W(1)),(\dot{\gamma}(0),-\dot{\gamma}(1))\rangle$. For $\mathrm{E}_*(W)$ to be zero $(W(0),W(1))$ must be perpendicular to $(\dot{\gamma}(0),-\dot{\gamma}(1))$. Since $f_*(v) = (W(0),W(1))$ for all $v \in T_xN$, we see that $E_*(W) = 0$ if and only if γ is a smooth geodesic and $f_*(T_xN) \perp (\dot{\gamma}(0),-\dot{\gamma}(1))$. □

We wish to do Morse Theory on the space $P(M,f)$. We will follow the approach used earlier and for a fixed $c > 0$ with $P_c(M,f) \neq \emptyset$ approximate $P_{<c}(M,f)$ up to homotopy equivalence by a finite dimensional manifold. We will approximate $P_{<c}(M,f)$ with the smooth manifold $B_c^k(M,f)$. However for that to work we must prove a theorem analogous to Theorem 1.24.

Theorem 3.7. *The map* $\mathrm{E}' = \mathrm{E}_{|B^k_c(M,f)} : B^k_c(M,f) \longrightarrow [0,c)$ *is smooth and for each* $a < c$ *the set* $\mathrm{E}'^{-1}([0,a]) = B_a(M,f)$ *is compact (*E' *is proper).* $B_a(M,f)$ *is a deformation retract of the corresponding set* $P_a(M,f)$ *and* $B^k_c(M,f)$ *is a deformation retract of* $P_{<c}(M,f)$*. Furthermore the critical points of* E' *are the same as the critical points of* E *in* $B^k_c(M,f)$*. The index of the Hessian of* E' *at a critical point* (x,γ) *is the same as the index of* E_{**} *at* (x,γ)*.*

Proof (Sketch): Recall that $B^k_c(M,f) = \{(x,\gamma) \in N \times B^k_c(M) \mid f(x) = (\gamma(0),\gamma(1))\}$. Since M is compact and complete we can use the set-up in Section 1.2 for $P(M)$. It is clear that E' is a smooth map since it is the composition of the projection onto the second factor and the energy function $\mathrm{E}' : B^k_c(M) \longrightarrow [0,c)$ which is smooth. Since the energy function $\mathrm{E}' : B^k_c(M) \longrightarrow [0,c)$ is proper and N is compact, E' is proper.

We have to see that there is a homotopy equivalence (deformation retraction) between the sets $P_{<c}(M,f)$ and $B^k_c(M,f)$. Since $P_{<c}(M,f) \subset N \times P_{<c}(M)$ and $B^k_c(M,f) \subset N \times P_{<c}(M)$ we can use a retraction which is the identity on the first factor. On the second factor, we can use the retraction r given in [M, proof of Theorem 16.2]. We will briefly recall the construction r.

Choose a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ such that $t_i - t_{i-1} < (\mathrm{inj}(M)^2)/c$. Let $r(\omega)$ denote the unique broken geodesic in $B^k_c(M)$ such that $r(\omega)_{|[t_{i-1},t_i]}$ is a geodesic of length less than $\mathrm{inj}(M)$ from $\omega(t_{i-1})$ to $\omega(t_i)$. Then one checks that $r(\omega)$ can be so defined and that $\mathrm{E}(r(\omega)) \leq \mathrm{E}(\omega) < c$. This retraction fits into a 1-parameter family of maps $r_s \colon P_{<c}(M) \longrightarrow P_{<c}(M)$ as follows. For $t_{i-1} \leq s \leq t_i$ let

$$\begin{cases} r_s(\omega)_{|[0,t_{i-1}]} = r(\omega)_{|[0,t_{i-1}]}, \\ r_s(\omega)_{|[t_{i-1},s]} = \text{minimal geodesic from } \omega(t_{i-1}) \text{ to } \omega(s), \\ r_s(\omega)_{|[s,1]} = \omega_{|[s,1]}. \end{cases}$$

Then r_0 is the identity map of $P_{<c}(M)$ and $r_1 = r$. One then proves that this retraction is continuous as a function of both variables. The retraction $P_{<c}(M,f) \longrightarrow B^k_c(M,f)$ given by $(x,\omega) \longmapsto (x,r(\omega))$ is then well-defined since $f(x) = (\omega(0),\omega(1)) = (r(\omega(0)), r(\omega(1)))$, i.e. $(x,r(\omega))$ is a point in $B^k_c(M,f)$. For $a < c$ this also gives the desired deformation retraction of $P_a(M,f)$ onto $B_a(M,f)$.

Since the critical points of E in $P_{<c}(M,f)$ are points (x,γ) where γ is a smooth geodesics, it is clear that E and E' have the same critical points. The argument that the indices are the same goes as in [M, proof of Theorem 16.2]. □

Remark. We note that the set $\mathrm{E}^{-1}(0) \subseteq P(M,f)$ is the set

$$\mathrm{E}^{-1}(0) = \{(x,\gamma) \in P(M,f) \mid f(x) = (\gamma(0),\gamma(1)),\ \gamma(t) = \gamma(0) = \gamma(1)\ \forall t \in [0,1]\}.$$

This set is naturally identified with $f^{-1}(\Delta)$, where $\Delta \subset M \times M$ is the diagonal submanifold. △

3.3 The Main Theorem and an Index Estimate

In this section we give a proof of the main theorem in [FMR]. For the proof of that theorem we need the following technical proposition which we do not prove. We will also estimate the index of a non-trivial critical point in terms of the asymptotic index of f.

Proposition 3.8. *Let M, N be as above and let $f\colon N \longrightarrow M\times M$ be an isometric immersion. Let $S = \{(x, f(x), \dots, f(x)) \mid x \in f^{-1}(\Delta)\}$. Then S is a deformation retract of an open neighbourhood in $N \times (M \times M) \times \cdots \times (M \times M)$ if one of the following conditions are satisfied*

(1) f is minimal,

(2) $N = N_1 \times N_1$ and $f = f_1 \times f_1$ where $f_1\colon N_1 \longrightarrow M$ is an embedding.

Proof: [FMR, Proposition 1.3]. □

We will now prove the main theorem of this section.

Theorem 3.9. *Let M, N be as above and let $f\colon N \longrightarrow M \times M$ be an isometric immersion. Assume that every non-trivial critical point of $\mathrm{E} : P(M, f) \longrightarrow \mathbb{R}$ has index $\geq \lambda$.*

(1) If $\lambda \geq 1$ then $f^{-1}(\Delta) \neq \emptyset$.

(2) If $\lambda \geq 2$ and M is simply connected then $f^{-1}(\Delta)$ is path connected.

If, in addition, f satisfies one of the conditions in Proposition 3.8 then

(3) $\pi_i(P(M, f), f^{-1}(\Delta)) = 0$ for all $i < \lambda$,

(4) there is an exact sequence of homotopy groups

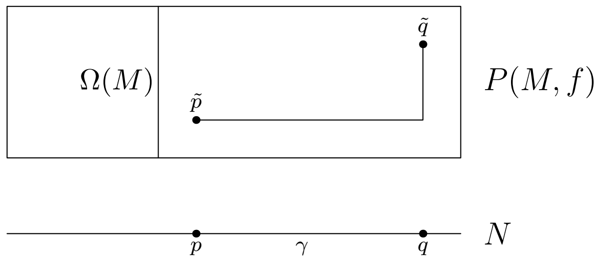
$$\longrightarrow \pi_i(f^{-1}(\Delta)) \longrightarrow \pi_i(N) \xrightarrow{\partial} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta)) \longrightarrow$$

where $i < \lambda$. The map ∂ is given by $(p_1 f)_* - (p_2 f)_*$ where p_1, p_2 are the projections of $M \times M$ onto the first and second factor, respectively.

Remark. (3) and (4) are not interesting if $\lambda = 0$. We will therefore assume that $\lambda \geq 1$ △

Proof: (1) If $f^{-1}(\Delta) = \mathrm{E}^{-1}(0) = \emptyset$ then E would have a minimum at some non-trivial critical point (x, γ). Hence the index of E at that point is zero, a contradiction to the assumption that every non-trivial critical point has index ≥ 1.
(2) Assume that $f^{-1}(\Delta)$ is not path connected, that is, $f^{-1}(\Delta) = A \cup B$ where $A \cap B = \emptyset$ and A, B closed. As above, we identify $f^{-1}(\Delta) = \mathrm{E}^{-1}(0) \subseteq P(M, f)$.

We first show that M simply connected implies that $P(M, f)$ is path connected. We note that $\pi\colon P(M, f) \longrightarrow N$ is a fibration and as such has the homotopy lifting property with respect to any topological space X. We will use this property for $X = *$. Since $\pi_i(\Omega(M)) = \pi_{i+1}(M)$ the fact that M is simply connected implies that $\Omega(M)$ is path connected. Let $\tilde{p}, \tilde{q} \in P(M, f)$ with $\pi(\tilde{p}) = p$ and $\pi(\tilde{q}) = q$. Since N is path connected we can choose a curve γ connecting p and q in N. Regard that map as a homotopy $\gamma\colon * \times I \longrightarrow N$. Lift γ to a curve in $P(M, f)$ connecting the two fibres $\pi^{-1}(p)$ and $\pi^{-1}(q)$. Since the fibres are path connected we get a curve from $\tilde{p}$ to $\tilde{q}$.



Let $p \in A$ and $q \in B$. As $P(M, f)$ is path connected we connect p and q with a curve in $P(M, f)$. Since the unit interval is compact we can find a $c > 0$ such that every point on that curve has energy less that c. We can assume that the curve is a curve connecting p and q in $B^k_c(M, f)$. Denote that curve $\bar{\gamma}$. The space $B^k_c(M, f)$ is a submanifold of the product $N \times M \times \cdots \times M$ and we endow $B^k_c(M, f)$ with the induced metric from that product. In order to finish the proof we need the following claim.

Claim 5. There exists a sequence of curves $\{\bar{\gamma}_k\}_{k=1}^{\infty}$ with $\bar{\gamma}_k \subseteq \mathrm{E}^{-1}([0, 1/k])$ such that $\bar{\gamma}_k(0) = p$ and $\bar{\gamma}_k(1) = q$. Furthermore $\bar{\gamma}_k$ is in the same homotopy class as $\bar{\gamma}$ for all k.

We can now finish the proof of (2). Let $d_{B^k_c(M,f)}$ denote the distance associated with the Riemannian metric on $B^k_c(M, f)$. Define a continuous function μ by $\mu(\sigma) = d_{B^k_c(M,f)}(\sigma, A) - d_{B^k_c(M,f)}(\sigma, B)$. By restricting μ to $\bar{\gamma}_k$ one get $\mu(p) < 0$ and $\mu(q) > 0$. Hence by continuity there exists a $\tau_k \in \bar{\gamma}_k$ such that $\mu(\tau_k) = 0$. By compactness and the Bolzano-Weierstrass property $\{\tau_k\}$ has a converging subsequence, so we can assume that $\tau_k \longrightarrow \tau$. Note also that by continuity $\mu(\tau) = 0$. Every point of $\bar{\gamma}_k$ has energy less than $1/k$, that is, $\mathrm{E}(\tau_k) \leq 1/k$. Since E is continuous $\mathrm{E}(\tau) = 0$. Now $\tau \in \mathrm{E}^{-1}(0)$ and $\mu(\tau) = 0$ which implies that $\tau \in A \cap B$, a contradiction.

A perhaps more natural argument is to say that all sublevels $\mathrm{E}([0, 1/k])$ are path connected and the intersection of all sublevels is $f^{-1}(\Delta)$ which is then also path connected. The proof of this is the same as the proof given above.

Proof of Claim (5): By [M, Corollary 6.8] we can choose a Morse function $\tilde{\mathrm{E}}$ such that $|\mathrm{E} - \tilde{\mathrm{E}}| \leq 1/10k$ on the compact sublevel $\{\sigma \in B^k_c(M, f)) \mid \tilde{\mathrm{E}}(\sigma) \leq c + \frac{1}{2}\}$ and such that all critical points of E in the set $\tilde{\mathrm{E}}([1/2k, c + \frac{1}{2}])$ has index at least 2. By Morse Theory

$\tilde{\mathrm{E}}^{-1}((-\infty, 1/2k])$ is homotopy equivalent to $\tilde{\mathrm{E}}^{-1}((-\infty, c+\frac{1}{2}])$ where cells of dimension at least 2 is attached. This implies that $\pi_1(\tilde{\mathrm{E}}^{-1}((-\infty, c+\frac{1}{2}]), \tilde{\mathrm{E}}^{-1}((-\infty, 1/2k])) = 0$. By using the long exact sequence for the pair $(\tilde{\mathrm{E}}^{-1}((-\infty, c+\frac{1}{2}]), \tilde{\mathrm{E}}^{-1}((-\infty, 1/2k]))$, we see that the induced map

$$i_*\colon \pi_1(\tilde{\mathrm{E}}^{-1}((-\infty, 1/2k])) \longrightarrow \pi_1(\tilde{\mathrm{E}}^{-1}((-\infty, c+\tfrac{1}{2}]))$$

is surjective. This means that $\bar{\gamma}$ is homotopic to a map $\bar{\gamma}_k$ keeping endpoints fixed. Away from the critical points the homotopy is given by moving $\bar{\gamma}$ along the flow lines of $\tilde{\mathrm{E}}$. △

(3) The following argument is analogous to the argument in Theorem 2.4 Claim 2. We will give the argument again to emphasize the similarities.

If we had known that $f^{-1}(\Delta)$ is a manifold (3) would have followed from Proposition 1.25 without any restrictions on f. However that need not be true, so we have to use Proposition 1.25 together with Proposition 3.8. According to Theorem 3.7 we pick a $c > 0$ such that $P_c(M, f) \neq \emptyset$ and approximate $P_c(M, f)$ up to homotopy equivalence by the manifold $B_c^k(M, f)$. We have to see that $f^{-1}(\Delta)$ has an open neighbourhood $U \subset B_c^k(M, f)$ such that $f^{-1}(\Delta)$ is a retract of U. As before, $B_c^k(M, f)$ is identified with an open submanifold of $N \times M \times \cdots \times M$ with k even (if not, add an extra breakpoint).

Let S be the set $S = \{(x, f(x), \ldots, f(x)) \mid x \in f^{-1}(\Delta)\}$. Under the identification of $B_c^k(M, f)$ with an open submanifold of $N \times M \times \cdots \times M$ the map $x \longmapsto (x, f(x), \cdots, f(x))$ and $(x, f(x), \cdots, f(x)) \longmapsto x$ gives a diffeomorphism between the set S and $f^{-1}(\Delta)$. Hence the existence of the desired neighbourhood follows from Proposition 3.8.

By Proposition 1.25 $\pi_i(B_c^k(M, f), f^{-1}(\Delta)) = 0$ for all $i < \lambda$, since by Theorem 3.7 E' is smooth and proper on the set $B_c^k(M, f)$ and since the index of the critical points of E are the same as the index of the critical points of E'. This means that up to homotopy $B_c^k(M, f)$ is obtained from $f^{-1}(\Delta)$ by attaching cells of dimension at least λ. From Theorem 3.7 we know that there is a homotopy equivalence between $B_c^k(M, f)$ and $P_{<c}(M, f)$. This means that for all c with $P_c(M, f) \neq \emptyset$ $P_{<c}(M, f)$ is homotopy equivalent to a relative CW-complex $f^{-1}(\Delta) \cup e^1 \cup e^2 \cup \cdots$ where the dimension of e^i is at least λ. For c small enough it is possible that there are no critical values between 0 and c. In that case we do not attach any cells.

Let $c_0 < c_1 < c_2 < \cdots$ be a sequence of positive real numbers with $\lim_{i \longrightarrow \infty} c_i = \infty$ and such that for all i the set $P_{c_i}(M, f) \neq \emptyset$. Consider the directed system $\{P_{<c_i}(M, f)\}$ with union $P(M, f)$. The space $P(M, f)$ is paracompact since it is a metric space, see [MS, Theorem of A. H. Stone], and every point of $P(M, f)$ lies in the interior of some $P_{<c_i}(M, f)$. Hence [M, Example A1] implies that $P(M, f)$ is the homotopy direct limit of $\{P_{<c_i}(M, f)\}$. Since $P_{<c_i}(M, f)$ is homotopy equivalent to a relative CW-complex $K_i = f^{-1}(\Delta) \cup e^{i,0} \cup e^{i,1} \cup \cdots$ where the dimension of $e^{i,j}$ is at least λ, we have an increasing sequence of CW-complexes $K_0 \subset K_1 \subset K_2 \subset \cdots$. Let K be the union of the K_is. [M, Example A2] implies that K is the homotopy direct limit of the directed system $\{K_i\}$.

As all the CW-complexes K_i are obtained from $f^{-1}(\Delta)$ by attaching cells of dimension at least λ, K is also obtained from $f^{-1}(\Delta)$ by attaching cells of dimension at least λ. Since for all i the maps $P_{<c_i}(M,f) \longrightarrow K_i$ are homotopy equivalences, Theorem 1.14 implies that the map $P(M,f) \longrightarrow K$ is a homotopy equivalence. This means that the relative homotopy groups

$$\pi_i(P(M,f), f^{-1}(\Delta)) = 0 \quad \text{for} \quad i < \lambda.$$

(4) By the proof of (3), $f^{-1}(\Delta)$ can be obtained from $P(M,f)$ by gluing cells of dimension at least $\lambda \geq 1$. The sets $f^{-1}(\Delta)$ and $P(M,f)$ have the same number of path connected components, since gluing cells of dimension greater than one does not change the number of path connected components. Hence the inclusion induces a bijection $i_*\colon \pi_0(f^{-1}(\Delta)) \longrightarrow \pi_0(P(M,f))$. From (3) we know that $\pi_i(P(M,f), f^{-1}(\Delta)) = 0$ for all $i < \lambda$. By the long exact sequence for the pair $(P(M,f), f^{-1}(\Delta))$ this is the same as $\pi_i(P(M,f)) \cong \pi_i(f^{-1}(\Delta))$ for $i < \lambda - 1$ and that the induced map $\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_i(P(M,f))$ is a surjection for $i = \lambda - 1$.

The commutative diagram of fibrations in Definition 3.2 gives rise to two exact sequences where all the squares commute

$$\begin{array}{ccccccccccc}
\pi_i(\Omega(M)) & \to & \pi_i(P(M,f)) & \longrightarrow & \pi_i(N) & \xrightarrow{\partial} & \pi_{i-1}(\Omega(M)) & \to & \pi_{i-1}(P(M,f)) & \longrightarrow & \pi_{i-1}(N) \\
\downarrow = & & \downarrow & & \downarrow f_* & & \downarrow = & & \downarrow & & \downarrow f_* \\
\pi_i(\Omega(M)) & \longrightarrow & \pi_i(P(M)) & \xrightarrow{p_*} & \pi_i(M\times M) & \xrightarrow{\tilde\partial} & \pi_{i-1}(\Omega(M)) & \longrightarrow & \pi_{i-1}(P(M)) & \xrightarrow{p_*} & \pi_{i-1}(M\times M)
\end{array}$$

Using the fact that $\pi_i(\Omega(M)) = \pi_{i+1}(M)$ we get another two exact sequences

$$\begin{array}{ccccccccccc}
\pi_{i+1}(M) & \longrightarrow & \pi_i(P(M,f)) & \longrightarrow & \pi_i(N) & \xrightarrow{\partial} & \pi_i(M) & \longrightarrow & \pi_{i-1}(P(M,f)) & \longrightarrow & \pi_{i-1}(N) \\
\downarrow = & & \downarrow & & \downarrow f_* & & \downarrow = & & \downarrow & & \downarrow f_* \\
\pi_{i+1}(M) & \longrightarrow & \pi_i(P(M)) & \xrightarrow{p_*} & \pi_i(M\times M) & \xrightarrow{\tilde\partial} & \pi_i(M) & \longrightarrow & \pi_{i-1}(P(M)) & \xrightarrow{p_*} & \pi_{i-1}(M\times M)
\end{array}$$

Exactness of the sequence

$$\longrightarrow \pi_i(f^{-1}(\Delta)) \longrightarrow \pi_i(N) \xrightarrow{\partial} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta)) \longrightarrow$$

for $i < \lambda - 1$ at the second and the fourth arrow follows from the previous diagram and the fact that $\pi_i(P(M,f)) \cong \pi_i(f^{-1}(\Delta))$ for $i < \lambda - 1$. For $i = \lambda - 1$ the exactness of the above sequence follows from the fact that the induced map $\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_i(P(M,f))$ is surjective. Note that since $f^{-1}(\Delta) \subset P(M,f)$ the map $\pi_i(M) \longrightarrow \pi_{i-1}(P(M,f))$ restricts to a map $\pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta))$.

The main point is therefore to see that $\partial = (p_1 f)_* - (p_2 f)_*$. Since $\partial = \tilde\partial f_*$ it is enough to show that $\tilde\partial$ is given by $p_{1*} - p_{2*}$. Consider the diagonal map $\varphi\colon M \longrightarrow M \times M$ given by

$\varphi(p) = (p, p)$. Using the fact that $\pi_i(M \times M)$ is canonically isomorphic to $\pi_i(M) \oplus \pi_i(M)$ we get a short exact sequence

$$0 \longrightarrow \pi_i(M) \xrightarrow{\varphi_*} \pi_i(M) \oplus \pi_i(M) \xrightarrow{\text{proj}_*} \pi_i(M) \longrightarrow 0$$

Since the sequence is exact, the map $\text{proj}_* : \pi_i(M) \oplus \pi_i(M) \longrightarrow \pi_i(M)$ is given by $p_{1*} - p_{2*}$ which proves (4). This ends the proof of the theorem. □

We wish to express the index of a critical point $(x, \gamma) \in P(M, f)$ in terms of the asymptotic index ν_f of the isometric immersion f. In order to do this we must see that the second variation formula of E can be rewritten so that the second fundamental form II' of f enters the formula. This is done in the following lemma.

Lemma 3.10. *Let* E, f *and* $P(M, f)$ *be as above. Let* (x, γ) *be a critical point of* E. *Let* (v, W) *be a tangent vector of* $P(M, f)$ *at* (x, γ) *with* W *a smooth and parallel vector field along* γ. *Then*

$$\frac{1}{2}\,\mathrm{E}_{**}(W, W) = -\int_0^1 \langle R(\dot\gamma, W)\dot\gamma, W\rangle dt + \langle II'(f_*(v), f_*(v)), (\dot\gamma(0), -\dot\gamma(1))\rangle.$$

Proof: Recall that $(W(0), W(1)) = f_*(v)$ for $(v, W) \in T_{(x,\gamma)}P(M, f)$. The second variation of E is given by

$$\frac{1}{2}\,\mathrm{E}_{**}(W, W) = -\sum_{t_i}\Big\langle W(t_i), \Delta_{t_i}\frac{DW}{dt}\Big\rangle - \int_o^1 \Big\langle W, \frac{D^2W}{dt^2} + R(\dot\gamma, W)\dot\gamma\Big\rangle dt.$$

Since W is parallel the term D^2W/dt^2 vanishes. The term $-\sum_{t_i}\langle W(t_i), \Delta_{t_i}(DW/dt)\rangle$ is only nonzero for $t_i = 0, 1$ since W is smooth. By regarding γ as a geodesic $(\gamma(t), \gamma(1-t))$ in the product $M \times M$ we get

$$-\Big\langle W(0), \frac{DW}{dt}(0)\Big\rangle - \Big\langle W(1), \frac{DW}{dt}(1)\Big\rangle = \langle (W(0), W(1)), \nabla_{\dot\gamma}(W(0), W(1))\rangle,$$

where $\nabla_{\dot\gamma}$ is covariant derivation with respect to the geodesic $(\gamma(t), \gamma(1-t))$. By symmetry of the connection we have

$$\langle (W(0), W(1)), \nabla_{\dot\gamma}(W(0), W(1))\rangle = \langle (W(0), W(1)), \nabla_{(W(0),W(1))}(\dot\gamma(0), -\dot\gamma(1))\rangle.$$

By using the definition of II' and the fact that $(W(0), W(1)) = f_*(v)$ we get

$$\langle (W(0), W(1)), \nabla_{(W(0),W(1))}(\dot\gamma(0), -\dot\gamma(1))\rangle = \langle II'(f_*(v), f_*(v)), (\dot\gamma(0), -\dot\gamma(1))\rangle,$$

which proves the lemma. □

Using this formula for E_{**} we can now formulate the relation between the index of a non-trivial critical point and the asymptotic index of the isometric immersion f.

Proposition 3.11. *Assume that M^m has positive lth Ricci curvature and let* E, f *and* $P(M,f)$ *be as above. Let* (x,γ) *be a critical point of* E *in* $P(M,f)$ *and let* λ *be the index of* (x,γ) *as a critical point. Then*

(1) $\lambda \geq \nu_f - m - l + 1$.

(2) If $f = f_1 \times f_2 \colon N_1 \times N_2 \longrightarrow M \times M$ *where* $f_i \colon N_i \longrightarrow M$ *is an immersion then* $\lambda \geq \nu_f - m - l + 2$.

Proof: (1) Let V denote the vector space of parallel vector fields along γ and perpendicular to $\dot{\gamma}$. Clearly V can be identified with the set $\{(W(0), W(1)) \mid W \in V\}$ and is as such a subspace of $T_{f(x)}M \times M$, $f(x) = (\gamma(0), \gamma(1))$. (Actually V is determined by the value of the vector fields at $t = 0$ by parallel translation, but we need the value at both $t = 0$ and $t = 1$ for the rest of the argument). Since V is orthogonal to $\dot{\gamma}$ we see, using the above identification, that V is orthogonal to $(\dot{\gamma}(0), -\dot{\gamma}(1))$.

Define a symmetric quadratic form A on V by $\langle A(v), v\rangle = -\int_0^1 \langle R(\dot{\gamma}, W)\dot{\gamma}, W\rangle dt$ where $v = (W(0), W(1))$ and $\langle -, - \rangle$ is the inner product induced from the standard metric on $T_{f(x)}M \times M$. Let V_1 be a subspace of V of maximal dimension such that $A_{|V_1}$ is negative definite. Regard V_1 as a subspace of $T_{f(x)}M \times M$.

Let N_x denote a maximal subspace of T_xN such that $II_{|N_x} = 0$. Since (x,γ) is a critical point $f_*(T_xN)$ is orthogonal to $(\dot{\gamma}(0), -\dot{\gamma}(1))$, hence $f_*(N_x)$ is orthogonal to $(\dot{\gamma}(0), -\dot{\gamma}(1))$.

We have $\mathrm{E}_{**}(W, W) = \langle A(v), v\rangle < 0$ for all $v \in f_*(N_x) \cap V_1$. Since f is an immersion we can estimate the dimension of $f_*(N_x) \cap V_1$ as follows $\dim(f_*(N_x) \cap V_1) \geq \nu_f(x) + \dim V_1 - 2m + 1$.

We conclude the proof of (1) by showing that $\dim V_1 \geq m - l$. Under the identification of V with $\{(W(0), W(1)) \mid W \in V\}$ pick an orthonormal basis $\{v_1, \dots, v_{m-1}\}$ of V consisting of eigenvectors of the quadratic form A. By using the assumption that M has positive lth Ricci curvature we get

$$\sum_{i=1}^{l} \langle A(v_i), v_i\rangle = \sum_{i=1}^{l} - \int_0^1 \langle R(\dot{\gamma}, v_i)\dot{\gamma}, v_i\rangle dt = -\int_0^1 \sum_{i=1}^{l} \langle R(\dot{\gamma}, v_i)\dot{\gamma}, v_i\rangle dt < 0.$$

This implies that there exists an i such that $\langle A(v_i), v_i\rangle < 0$. Assume without loss of generality that $i = 1$. Similarly $\sum_{i=2}^{l+1} \langle A(v_i), v_i\rangle < 0$ so again there exists an i such that $\langle A(v_i), v_i\rangle < 0$. Assume $i = 2$. Continue like this until we have picked $\{v_1, \dots, v_{m-l}\}$ such that $\langle A(v_i), v_i\rangle < 0$ for all $i = 1, \dots, m - l$. This implies that $\langle A(v), v\rangle < 0$ for all $v \in \mathrm{span}\{v_1, \dots, v_{m-l}\}$ proving that $\dim V_1 \geq m - l$. Since this is true for an arbitrary x we have proved (1).

(2) Let $f = f_1 \times f_2 \colon N_1 \times N_2 \longrightarrow M \times M$ where $f_i \colon N_i \longrightarrow M$ is an immersion. Let (x, γ), $x = (x_1, x_2)$, be a critical point. Since (x, γ) is a critical point $f_*(T_x N)$ is perpendicular to $(\dot\gamma(0), -\dot\gamma(1))$. Hence γ is a geodesic such that $f_{1*}(T_{x_1} N_1)$ is perpendicular to $\dot\gamma(0)$ and $f_{2*}(T_{x_2} N_2)$ is perpendicular to $\dot\gamma(1)$. This implies that the linearly independent vectors $(\dot\gamma(0), 0)$ and $(0, \dot\gamma(1))$ are both perpendicular to $f_*(N_x)$ and V_1. Hence by using the proof of (1) we get $\dim(f_*(N_x) \cap V_1) \geq \nu_f(x) + \dim(V_1) - 2m + 2$. The proof is finished in the same way as in (1). □

An immediate consequence of Theorem 3.9 and Proposition 3.11 is the following.

Corollary 3.12. *Let M^m be a simply connected manifold of positive sectional curvature. Let $f \colon N^n \longrightarrow M^m$ be an isometric immersion. If $\nu_f > m/2$ then f must be an embedding.*

Proof: Since f is an immersion it is enough to show that f is a one-to-one map. Consider the isometric immersion $f_1 = f \times f \colon N \times N \longrightarrow M \times M$. Theorem 3.9 (2) together with Proposition 3.11 (2) implies that $f_1^{-1}(\Delta)$ is path connected for $\nu_{f_1} = 2\nu_f > m$ and hence for $\nu_f > m/2$. Since

$$f_1^{-1}(\Delta) = \{(x, x) \in N \times N\} \cup \{(x, y) \in N \times N \mid f(x) = f(y),\ x \neq y\}$$

we see that since $f_1^{-1}(\Delta)$ is path connected, f is injective. □

As another consequence of Theorem 3.9 and Proposition 3.11 we also state a slightly more general version of Frankel's theorem where the theorem is formulated in terms of isometric immersions and the asymptotic index.

Proposition 3.13. *Assume that M^m has positive sectional curvature. Let $f_i \colon N_i \longrightarrow M$ $i = 1, 2$ be an isometric immersion with asymptotic index ν_{f_i}. If $\nu_{f_1} + \nu_{f_2} \geq m$ then $f_1(N_1) \cap f_2(N_2) \neq \emptyset$.*

Proof: Consider the isometric immersion $f = f_1 \times f_2 \colon N_1 \times N_2 \longrightarrow M \times M$. By Theorem 3.9 (1) and Proposition 3.11 (2) for $l = 1$ we know that $f^{-1}(\Delta) \neq \emptyset$ as $\nu_f = \nu_{f_1} + \nu_{f_2} \geq m$. This clearly implies that $f_1(N_1) \cap f_2(N_2) \neq \emptyset$. □

3.4 A Generalization of Wilking's Theorem

In this section we will prove a generalization of Wilking's theorem, Theorem 2.4. We will first state the theorems and lemmas under the assumption that M has positive sectional curvature and then indicate the few necessary changes in the proofs to generalize the statements to the case where M has positive lth Ricci curvature.

Theorem 3.14. *Assume that M^m has positive sectional curvature. Let N_j, $j = 1, 2$ be embedded submanifolds with asymptotic index ν_j. Then*

(1) the inclusion $i_j \colon N_j \longrightarrow M$ *is* $(2\nu_j - m + 1)$*-connected.*

(2) if N_j *are both minimal (the inclusion is minimal),* $\nu_1 + \nu_2 \geq m$ *and* $\nu_2 \geq \nu_1$ *then* $i \colon N_1 \cap N_2 \longrightarrow N_1$ *is* $(\nu_1 + \nu_2 - m)$*-connected.*

Remark. Remember, that M is compact follows from the fact that M has positive sectional curvature. △

Remark. If N_j is a totally geodesic embedded submanifold then $\nu_j = \dim N_j$ and since $2\nu_j = 2(\dim M - \operatorname{codim} N)$ we get $2\nu_2 - m + 1 = m - 2 \operatorname{codim} N_j + 1$. Hence we see that Theorem 3.14 (1) reduces to the statement in Theorem 2.4 (1). Since a totally geodesic embedding is also minimal, we see that Theorem 2.4 (2) also follows from Theorem 3.14 (2). △

For the proof of this theorem we need a couple of lemmas.

Lemma 3.15. *Assume that* M^m *has positive sectional curvature. Let* $f = f_1 \times f_2 \colon N = N_1 \times N_2 \longrightarrow M \times M$ *be an isometric immersion and let* ν_f *be the asymptotic index of* f*. Assume either that the map* f *is a minimal immersion or* $N_1 = N_2$ *and* $f_1 = f_2$*, with* $f_1 \colon N_1 \longrightarrow M$ *an embedding. Then there are natural isomorphisms (bijections for* $i = 1$*)*

$$\pi_i(N_1, f^{-1}(\Delta)) \longrightarrow \pi_i(M, N_2) \quad \text{and} \quad \pi_i(N_2, f^{-1}(\Delta)) \longrightarrow \pi_i(M, N_1)$$

for $i \leq \nu_f - m$*. For* $i = \nu_f - m + 1$ *the maps are surjections.*

Remark. Since $f^{-1}(\Delta)$ is not a subset of N_1 the symbol $\pi_i(N_1, f^{-1}(\Delta))$ does not a priori make sense. However, the following general construction gives $\pi_i(N_1, f^{-1}(\Delta))$ meaning. Let $f \colon X \longrightarrow Y$ be a continuous map between topological spaces. Define the mapping cylinder M_f of f as the quotient space of the disjoint union $(X \times I) \amalg Y$ obtained by identifying each $(x, 1) \in X \times I$ with $f(x) \in Y$. The topology on M_f is the quotient topology and M_f is canonically homotopy equivalent to Y. Define $\pi_i(Y, X) = \pi_i(M_f, X)$.

This construction is functorial so we get a diagram

$$\longrightarrow \pi_i(X) \longrightarrow \pi_i(M_f) \longrightarrow \pi_i(M_f, X) \longrightarrow$$

We will always write this diagram as

$$\longrightarrow \pi_i(X) \longrightarrow \pi_i(Y) \longrightarrow \pi_i(Y, X) \longrightarrow$$

The homotopy groups $\pi_i(N_j, f^{-1}(\Delta))$ is defined in this way via the map $f^{-1}(\Delta) \longrightarrow N_1 \times N_2 \longrightarrow N_j$ where the first map is the inclusion and the second map is the projection on the jth factor. If f_j is only an immersion, the groups $\pi_i(M, N_j)$ are also constructed in this way using the map f_j. △

Proof (Lemma 3.15): Consider the fibration $P(M,f) \longrightarrow N_1 \times N_2$. By composing with the projection onto N_1 we get a fibration $P(M,f) \longrightarrow N_1$ with fibre V. We also have a fibration $\Omega(M) \longrightarrow P(M,*) \longrightarrow M$ and a map $f_2\colon N_2 \longrightarrow M$. By picking a basepoint in N_1 and writing the definition of $f_2^* P(M,*)$ explicitly, one sees that $f_2^* P(M,*) = V$. Hence we get a commutative diagram of fibrations

$$\begin{array}{ccccc} \Omega(M) & \longrightarrow & V & \longrightarrow & N_2 \\ \downarrow {\scriptstyle =} & & \downarrow & & \downarrow {\scriptstyle f_2} \\ \Omega(M) & \longrightarrow & P(M,*) & \longrightarrow & M \end{array}$$

Since $P(M,*)$ is contractible V is the homotopy fibre of the map $f_2\colon N_2 \longrightarrow M$. If f_2 is an embedding it follows from [H, p 407] that $\pi_{i+1}(M,N_2)$ and $\pi_i(V)$ can be identified for all i. If f_2 is only an immersion we have to use a mapping cylinder argument. Let M_{f_2} be the mapping cylinder of f_2 and let $g\colon N_2 \longrightarrow M_{f_2}$ be the induced map. Let A be the homotopy fibre of g. We convert the two maps to fibrations. Using the linear retraction $M_{f_2} \longrightarrow M$ we see that there is a map $A \longrightarrow V$. By using the two long exact sequences for the two fibrations we see that the map $A \longrightarrow V$ is a weak homotopy equivalence. Since $N_2 \subset M_{f_2}$, [H, p 407] implies that $\pi_{i+1}(M,N_2)$ and $\pi_i(V)$ can be identified for all i.

Using the fact that $\pi_i(V) \cong \pi_{i+1}(M,N_2)$, the long exact homotopy sequence for the fibration $V \longrightarrow P(M,f) \longrightarrow N_1$ and the long exact homotopy sequence for the pair $(N_1, f^{-1}(\Delta))$ (the map $f^{-1}(\Delta) \longrightarrow N_1$), we get a diagram

$$\begin{array}{ccccccccc} \longrightarrow & \pi_{i+1}(N_1, f^{-1}(\Delta)) & \longrightarrow & \pi_i(f^{-1}(\Delta)) & \longrightarrow & \pi_i(N_1) & \longrightarrow & \pi_i(N_1, f^{-1}(\Delta)) & \longrightarrow \\ & \downarrow {\scriptstyle f_{1*}} & & \downarrow & & \downarrow {\scriptstyle =} & & \downarrow {\scriptstyle f_{1*}} & \\ \longrightarrow & \pi_{i+1}(M,N_2) & \longrightarrow & \pi_i(P(M,f)) & \longrightarrow & \pi_i(N_1) & \longrightarrow & \pi_i(M,N_2) & \longrightarrow \end{array}$$

Since we have assumed that f is either a minimal immersion or $f = f_j \times f_j\colon N = N_j \times N_j \longrightarrow M \times M$ with f_j an embedding, we get by Theorem 3.9 (3) and Proposition 3.11 (2) that $\pi_i(P(M,f), f^{-1}(\Delta)) = 0$ for $i < \nu_f - m + 1$. By using the long exact homotopy sequence for the pair $(P(M,f), f^{-1}(\Delta))$ this is the same as the map

$$i_*\colon \pi_i(f^{-1}(\Delta)) \longrightarrow \pi_i(P(M,f))$$

is an isomorphism (bijection for $i = 0$) for $i < \nu_f - m$ and a surjection for $i = \nu_f - m$. Hence by the Five-Lemma we get immediately that f_{1*} is an isomorphism for $i < \nu_f - m$ and a surjection for $i = \nu_f - m$. Using the fact that f_{1*} is a surjection for $i = \nu_f - m$ and the Five-Lemma one can actually see that f_{1*} is an isomorphism for $i = \nu_f - m$ and a surjection for $i = \nu_f - m + 1$.

The second statement follows by switching N_1 and N_2 in the above proof. □

Lemma 3.16. *Assume that M^m has positive sectional curvature. Let N_j, $j = 1, 2$ be embedded submanifolds with asymptotic index $\nu_j = \nu_{i_j}$. If N_1 and N_2 are minimal or $N_1 = N_2$ there are natural isomorphisms*

$$\pi_i(N_1, N_1 \cap N_2) \longrightarrow \pi_i(M, N_2) \quad \text{and} \quad \pi_i(N_2, N_1 \cap N_2) \longrightarrow \pi_i(M, N_1)$$

for $i \leq \nu_1 + \nu_2 - m$. For $i = \nu_1 + \nu_2 - m + 1$ the maps are surjections.

Proof: If $f = f_1 \times f_2$ we clearly have $\nu_f = \nu_{f_1} + \nu_{f_2}$. If we put $f_1 = i_1$ and $f_2 = i_2$ it is also clear that $f^{-1}(\Delta) = N_1 \cap N_2$. By using the previous lemma we get the desired result. □

Proof (Theorem 3.14): The theorem contains no new information if $2\nu_j - m + 1 < 1$, so assume $2\nu_j - m + 1 \geq 1$.
(1) The inclusion induces a bijection $i_*\colon \pi_0(N_j) \longrightarrow \pi_0(M)$, since both N_1, N_2 and M are path connected. Consider the case $N_1 = N_2$. Since $\pi_i(N_2, N_2) = 0$ we get from the previous lemma that $\pi_i(M, N_2) = 0$ for $i \leq \nu_2 + \nu_2 - m + 1 = 2\nu_2 - m + 1$. Hence the desired result follows from the long exact homotopy sequence for the pair (M, N_2).
(2) By Proposition 3.13 $N_1 \cap N_2 \neq \emptyset$, since $\nu_1 + \nu_2 - m \geq 0$. Since both N_1 and N_2 are path connected, $N_1 \cap N_2$ is path connected. Hence the inclusion induces a bijection $i_*\colon \pi_0(N_1 \cap N_2) \longrightarrow \pi_0(N_1)$.

Since $\nu_2 \geq \nu_1$ we know that $\pi_i(M, N_2) = 0$ for $i \leq \nu_1 + \nu_2 - m$ by (1). Also, as N_1 and N_2 are minimal we know that $\pi_i(N_1, N_1 \cap N_2) \cong \pi_i(M, N_2)$ for $i \leq \nu_1 + \nu_2 - m$ by the previous lemma. Hence, by using the long exact homotopy sequence for the pair $(N_1, N_1 \cap N_2)$, (2) follows from the fact that $\pi_i(N_1, N_1 \cap N_2) = 0$ for $i \leq \nu_1 + \nu_2 - m$. □

Remark. The generalization to positive lth Ricci curvature is easy since the only place the curvature assumption enters are in the index estimates. Hence if M has positive lth Ricci curvature the map in (1) in Theorem 3.14 is $(2\nu_j - m + 2 - l)$-connected. The map in (2) is $(\nu_1 + \nu_2 - m + 1 - l)$-connected. △

Chapter 4

Totally Geodesic Submanifolds

In the previous chapters we studied the connectedness principle for positively curved manifolds. In this chapter we will use these results to investigate the homotopy theoretical and cohomological properties of manifolds with totally geodesic submanifolds. These results are also due to Wilking.

In this chapter we will only present a few of the consequences of Theorem 2.4 and Theorem 2.6. Many more can be found in Wilking's two articles [W I] and [W II].

We will throughout this chapter let p denote an arbitrary prime and assume that all manifolds and submanifolds are compact, path connected and oriented. All submanifolds are also assumed to be embedded.

In many cases the manifolds have positive sectional curvature, so the compactness assumption is in these cases automatic. In most cases the manifold will also be simply connected in which case it is also orientable.

4.1 Technical Results

We start by proving some basic topological statements. Some of these will be applied in the next section.

Lemma 4.1. *Let X be a simply connected CW-complex. Assume that X*

(1) has the integral cohomology ring of S^n, then X is homotopy equivalent to S^n.

(2) has the integral cohomology ring of $\mathbb{CP}^n$, then X is homotopy equivalent to $\mathbb{CP}^n$.

Proof: (1) The Universal Coefficient Theorem implies that X is also a homology sphere.

By the Hurewicz theorem we see that $\pi_i(X) = 0$ for $i = 1, \ldots, n-1$ and $\pi_n(X) = H_n(X;\mathbb{Z}) = \mathbb{Z}$. Choose a generator $f\colon S^n \longrightarrow X$. The induced map

$$f_*\colon \pi_n(S^n) \longrightarrow \pi_n(X)$$

maps $[\mathbb{1}_{S^n}] \longmapsto [f]$. By Hurewicz's Theorem the following diagram is commutative

$$\begin{array}{ccc} \pi_n(S^n) & \xrightarrow{f_*} & \pi_n(M) \\ h \downarrow \cong & & h \downarrow \cong \\ H_n(S^n;\mathbb{Z}) & \xrightarrow{f_*} & H_n(M;\mathbb{Z}) \end{array}$$

which implies that the induced map

$$f_*\colon H_n(S^n;\mathbb{Z}) \longrightarrow H_n(X;\mathbb{Z})$$

is an isomorphism. Hence Whitehead's Theorem implies that f is a homotopy equivalence.
(2) By [H, Theorem 4.57], the functor $H^2(-;\mathbb{Z})$ is represented by the Eilenberg MacLane space $K(\mathbb{Z},2)$. That is, for any CW-complex X there is an isomorphism $\langle X, K(\mathbb{Z},2)\rangle \cong H^2(X;\mathbb{Z})$ where $\langle X, K(\mathbb{Z},2)\rangle$ is the group of basepoint preserving maps from X into the Eilenberg MacLane space $K(\mathbb{Z},2)$.

Since $\mathbb{CP}^\infty$ is a model for $K(\mathbb{Z},2)$, see [H, Chapter 4], we know that the cohomology ring $H^*(K(\mathbb{Z},2);\mathbb{Z})$ is a polynomial ring $\mathbb{Z}[\beta]$, with β a generator in $H^2(K(\mathbb{Z},2);\mathbb{Z})$. Let $\alpha \in H^2(X;\mathbb{Z})$ be a generator and represent α by a map $f\colon X \longrightarrow K(\mathbb{Z},2)$. Then f^* maps a generator β of $H^2(K(\mathbb{Z},2);\mathbb{Z})$ to a generator α of $H^2(X;\mathbb{Z})$. The cohomology ring $H^*(X;\mathbb{Z})$ has the structure of a truncated polynomial ring $\mathbb{Z}[\alpha]/(\alpha^{n+1})$. Since f maps generator to generator the map f induces an isomorphism $f^*\colon H^i(K(\mathbb{Z},2);\mathbb{Z}) \longrightarrow H^i(X;\mathbb{Z})$, $i \leq 2n+1$.

By the Cellular Approximation Theorem $f\colon X \longrightarrow \mathbb{CP}^\infty$ is homotopic to a cellular map, which we also denote by f. Hence f defines a map from X to the $2n$-skeleton of $\mathbb{CP}^\infty$ which is $\mathbb{CP}^n$. Since the cohomology ring of $\mathbb{CP}^n$ is the cohomology ring of $\mathbb{CP}^\infty$ truncated at degree $2n+1$ the map $f\colon X \longrightarrow \mathbb{CP}^n$ induces an isomorphism in cohomology. Hence by the theorem of Whitehead, f is a homotopy equivalence. □

Remark. It is tempting to claim that $\mathbb{HP}^\infty = K(\mathbb{Z},4)$ and use the same proof to prove that a simply connected CW-complex with the same cohomology ring as $\mathbb{HP}^n$ is homotopy equivalent to $\mathbb{HP}^n$. It is, however, false that $\mathbb{HP}^\infty = K(\mathbb{Z},4)$! It is in fact also false that a space with the same cohomology ring as $\mathbb{HP}^n$ is homotopy equivalent to $\mathbb{HP}^n$. There are 6 distinct homotopy types of spaces with cohomology ring equal to the cohomology ring of $\mathbb{HP}^2$! △

Proposition 4.2. *Let M^m be a simply connected manifold. Let N^{m-k} be a submanifold such that the inclusion is $(m-k)$-connected. If k divides m, M has the integral cohomology ring of either S^m, $\mathbb{CP}^{m/2}$, $\mathbb{HP}^{m/4}$ or $Ca\mathbb{P}^2$.*

Remark. The list S^m, $\mathbb{CP}^{m/2}$, $\mathbb{HP}^{m/4}$ and $Ca\mathbb{P}^2$ are the simply connected compact rank one symmetric spaces (CROSS).

We note that the structure of the Cayley projective plane $Ca\mathbb{P}^2 = \mathsf{F}_4/\mathsf{Spin}(9)$ as a CW-complex is a 0-cell with an 8-cell and a 16-cell attached. The 16-cell is attached to the 8-cell via the Hopf map $\eta\colon S^{15} \longrightarrow S^7$. △

Proof: The strategy is to use cohomology with $\mathbb{Z}/p\mathbb{Z}$-coefficients and then use the Universal Coefficient Theorem to finish the proof.

Let e be the Poincare dual of the image of the fundamental class of N in $H_*(M;\mathbb{Z}/p\mathbb{Z})$. Then we know by Theorem 2.6 that the map $\cup e\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+k}(M;\mathbb{Z}/p\mathbb{Z})$ is an isomorphism for $0 < i < m-k$, injective for $i = m-k$ and surjective for $i = 0$. Since M is simply connected, M is connected which implies that $H^0(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. Since the map $\cup e$ is surjective for $i = 0$ we see that $H^k(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ or $H^k(M;\mathbb{Z}/p\mathbb{Z}) = 0$.

Assume first that $H^k(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. Since M is simply connected $H^1(M;\mathbb{Z}/p\mathbb{Z}) = 0$. Corollary 1.12 implies that $H^m(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ and $H^{m-1}(M;\mathbb{Z}/p\mathbb{Z}) = 0$.

Let $0 < q < k$. We claim that there is an isomorphism $H^q(M;\mathbb{Z}/p\mathbb{Z}) \cong H^{k-q}(M;\mathbb{Z}/p\mathbb{Z})$. To prove this, let $c \in H^q(M;\mathbb{Z}/p\mathbb{Z})$. By Theorem 2.6 this class is mapped isomorphically to a class in $H^{m-k+q}(M;\mathbb{Z}/p\mathbb{Z})$. By Poincare Duality this class is mapped isomorphically to a class in $H^{k-q}(M;\mathbb{Z}/p\mathbb{Z})$. This could also be formulated as the pairing $H^{k-q}(M;\mathbb{Z}/p\mathbb{Z}) \times H^q(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathbb{Z}/p\mathbb{Z}$ given by $(\alpha,\beta) \longmapsto (\alpha\cup\beta)(e)$ is nondegenerate. We will also call this pairing, or isomorphism, Poincare Duality.

Now we know that $H^k(M;\mathbb{Z}/p\mathbb{Z}) = H^{2k}(M;\mathbb{Z}/p\mathbb{Z}) = \cdots = H^m(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ with $H^k(M;\mathbb{Z}/p\mathbb{Z})$ generated by e, $H^{2k}(M;\mathbb{Z}/p\mathbb{Z})$ by e^2 etc. Poincare Duality implies that $e^{m/k}$ generates $H^m(M;\mathbb{Z}/p\mathbb{Z})$. Furthermore, by Poincare Duality and Theorem 2.6 the cohomology is periodic with period k. Since M is simply connected the period is at least 2.

Let q be the first degree such that $H^q(M;\mathbb{Z}/p\mathbb{Z}) \neq 0$, $1 < q \leq k$. Let $a \neq 0$ be an element in $H^q(M;\mathbb{Z}/p\mathbb{Z})$. The map $\cup a\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+q}(M;\mathbb{Z}/p\mathbb{Z})$ is injective. To prove this, we see that by Poincare Duality there exists a class $b \in H^{k-q}(M;\mathbb{Z}/p\mathbb{Z})$ such that $a \cup b = e$. Let $\alpha \in H^i(M;\mathbb{Z}/p\mathbb{Z})$, $0 < i < k$. Then if $\alpha \cup a = 0$ we see that $\alpha \cup e = 0$, hence $\alpha = 0$ since $\cup e$ is an isomorphism.

Next we will see that there exists a j such that $a^j = \gamma e$, $\gamma \neq 0$. As the map $\cup a$ is injective then $a^j \neq 0$ for all j. Let c be such that $cq > k$ and $(c-1)q < k$. Assume that $a^{cq} \neq 0 \in H^{cq}(M;\mathbb{Z}/p\mathbb{Z})$. As the cohomology is periodic with period k we see that $a^{cq \,(\mathrm{mod}\, k)} \neq 0 \in H^{cq\,(\mathrm{mod}\, k)}(M;\mathbb{Z}/p\mathbb{Z})$. This is a contradiction since $H^1(M;\mathbb{Z}/p\mathbb{Z}) = \cdots = H^{q-1}(M;\mathbb{Z}/p\mathbb{Z}) = 0$. Hence there exists a j such that $a^j \in H^k(M;\mathbb{Z}/p\mathbb{Z})$. Since $H^k(M;\mathbb{Z}/p\mathbb{Z})$ is one dimensional it follows that $a^j = \gamma e$.

We need to argue that $H^q(M;\mathbb{Z}/p\mathbb{Z})$ is one dimensional. Assume on the contrary that there exists two linearly independent classes $a, b \in H^q(M;\mathbb{Z}/p\mathbb{Z})$. By the argument above, the maps $\cup a$ and $\cup b$ are injective. The maps $\cup a^j, \cup b^j\colon H^q(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{k+q}(M;\mathbb{Z}/p\mathbb{Z})$ are also given by $\cup e\colon H^q(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{k+q}(M;\mathbb{Z}/p\mathbb{Z})$ which is an isomorphism. Hence $\cup a, \cup b$ are isomorphisms. By the above argument, $a^j = \gamma_1 e$ and $b^j = \gamma_2 e$. Since the maps $\cup a$ and $\cup b$ are injective, a and b are linearly dependent, which is a contradiction.

Assume next that $H^k(M;\mathbb{Z}/p\mathbb{Z}) = 0$. We know that $H^m(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. As above, according to Theorem 2.6 and Poincare Duality the cohomology groups are periodic with period k. But since the class $e \in H^k(M;\mathbb{Z}/p\mathbb{Z})$ is zero, the map

$$\cup e\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+k}(M;\mathbb{Z}/p\mathbb{Z})$$

is the zero map. But it is also an isomorphism. Hence $H^i(M;\mathbb{Z}/p\mathbb{Z}) = 0$ for $i = 1,\ldots,m-k$. By Poincare Duality $H^i(M;\mathbb{Z}/p\mathbb{Z}) = 0$ for $i = m-k,\ldots,m-1$.

Now we know that the cohomology groups $H^i(M;\mathbb{Z}/p\mathbb{Z})$ are

$$H^i(M;\mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & i = 0, q, \ldots, m \\ 0 & \text{otherwise} \end{cases}$$

or

$$H^i(M;\mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & i = 0, m \\ 0 & \text{otherwise.} \end{cases}$$

By the above argument we also know that the cup product structure of $H^*(M;\mathbb{Z}/p\mathbb{Z})$ is that of a truncated polynomial ring generated by a class in degree q or m. As p was arbitrary, this is true for all prime numbers. Note also that q does not depend on p.

We finish the proof by showing that the integral cohomology ring is also a truncated polynomial ring. For future reference we will state this as a claim.

Claim 6. Let M, p, q, m and $H^*(M;\mathbb{Z}/p\mathbb{Z})$ be as above. Then the integral cohomology ring of M is a truncated polynomial ring generated by a class in degree q or m.

Proof of Claim: Since the cohomology of M is concentrated in degrees $q, 2q, \ldots, m$ and $q \geq 2$ it follows from the Universal Coefficient Theorem that the integral cohomology groups contain no p-torsion. Furthermore, since the dimension of $H^i(M;\mathbb{Z}/p\mathbb{Z})$ as $\mathbb{Z}/p\mathbb{Z}$-vector space is 1 for $i = 0, q, 2q, \ldots, m$ and 0 otherwise, the rank of $H^*(M;\mathbb{Z})$ is 1 if $i = 0, q, 2q, \ldots m$ and 0 otherwise.

By functoriality, the ring homomorphism $\pi\colon \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$ induces a ring homomorphism $H^*(M;\mathbb{Z}) \longrightarrow H^*(M;\mathbb{Z}/p\mathbb{Z})$. This homomorphism is called reduction modulo p.

We must see that the cup product structure of $H^*(M;\mathbb{Z})$ is the same as the cup product structure of $H^*(M;\mathbb{Z}/p\mathbb{Z})$, namely a truncated polynomial ring generated by an element of degree either q or m. For that, let $a \in H^q(M;\mathbb{Z}) = \mathbb{Z}$ be a generator, i.e. $a = \pm 1$. Note that if a is reduced modulo an arbitrary prime p then $a \neq 0$, i.e. a generates $H^*(M;\mathbb{Z}/p\mathbb{Z})$ for all primes p. We claim that a^2 generates $H^{2q}(M;\mathbb{Z}) = \mathbb{Z}$. Assume on the contrary that $a^2 = d \neq \pm 1$. Write d as a product of primes $d = p_1^{r_1} \cdots p_k^{r_k}$ and note that if we reduce a^2 modulo one of the primes p_i then $a^2 = 0$. This is a contradiction since $a \in H^q(M;\mathbb{Z}/p_i\mathbb{Z})$ generates $H^*(M;\mathbb{Z}/p_i\mathbb{Z})$. Hence $H^{2q}(M;\mathbb{Z})$ is generated by a^2. By repeating the argument, we conclude that a generates $H^*(M;\mathbb{Z})$. △

Adams' Hopf Invariant One Theorem, see [A], implies that $q = 2, 4, 8$, since M is simply connected. If $q = 8$, then one can use Steenrod reduced powers at the prime 3 and the Adem relations, see [B, Theorem 15.20], to prove that $m = 16$. This ends the proof. □

Proposition 4.3. *Assume that M^m is n-connected ($1 \leq n < m-k$) and that N^{m-k} is a submanifold such that $i\colon N \longrightarrow M$ is $(m-k)$-connected. Assume furthermore that $k \mid m-n$. Then M and N are homotopy equivalent (homeomorphic if the dimensions are different[1] from 3) to a sphere.*

Proof: Again, we will first use $\mathbb{Z}/p\mathbb{Z}$-coefficients and use the Universal Coefficient Theorem to derive the result for $\mathbb{Z}$-coefficients. As M is n-connected and compact we know by the Cellular Approximation Theorem that M is a CW-complex with one zero cell and a finite number of cells all of dimension at least $n+1$. By cellular cohomology and Poincare Duality we know that

$$H^i(M;\mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & i=0 \\ 0 & i=1,2,\dots n \\ 0 & i=m-n, m-n+1, \dots m-1 \\ \mathbb{Z}/p\mathbb{Z} & i=m. \end{cases}$$

Since N is path connected we see that $H^0(N;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ and by Poincare Duality for N we also know that $H^{m-k}(N;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.

As the inclusion $i\colon N \longrightarrow M$ is $(m-k)$-connected Theorem 2.6 implies that the map $\cup e\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+k}(M;\mathbb{Z}/p\mathbb{Z})$ is an isomorphism for $0 < i < m-k$, surjective for $i=0$ and injective for $i = m-k$.

Since $k \mid m-n$ we can consider the map $\cup e^{(m-n)/k}\colon H^k(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{m-n}(M;\mathbb{Z}/p\mathbb{Z})$ which by Theorem 2.6 is an isomorphism. The fact that $H^{m-n}(M;\mathbb{Z}/p\mathbb{Z}) = 0$ implies that $H^k(M;\mathbb{Z}/p\mathbb{Z}) = 0$ which again implies that $e=0$. This implies by Theorem 2.6 that $H^i(M;\mathbb{Z}/p\mathbb{Z}) = 0$ for $i = n, n+1, \dots m-n-1$. Hence we see that M is a $\mathbb{Z}/p\mathbb{Z}$-cohomology sphere. By the Universal Coeficient Theorem, this is also true for cohomology with $\mathbb{Z}$-coefficients. Since $i\colon N \longrightarrow M$ is $(m-k)$-connected we conclude that N is also a cohomology sphere. To finish the proof we refer to Lemma 4.1 which implies that M and N are homotopy equivalent to a sphere. If the dimensions are different from 3 the generalized Poincare Conjecture, see [Sm] and [Fr], implies that M and N are homeomorphic to a sphere. □

4.2 Topology of Manifolds with Totally Geodesic Submanifolds

In this section we will use the technical results of the preceding section, Theorem 2.4 and Theorem 2.6 to derive some rather strong results on the topology of manifolds with totally geodesic submanifolds.

Proposition 4.4. *Assume that M^m has positive sectional curvature and contains two totally geodesic submanifolds, $N_i^{m-k_i}$, $i=1,2$.*

[1] It remains to be seen whether or not the work of Perelman makes this assumption superfluous.

(1) Suppose N_2 is homotopy equivalent to S^{m-k_2} and the inclusion map $i\colon N_2 \longrightarrow M$ is (k_1+1)-connected. Assume that $k_1 + 1 \le m - k_2$ or $k_2 = 0$ then M is homotopy equivalent to S^m.

(2) Suppose N_2 is homotopy equivalent to $\mathbb{CP}^{(m-k_2)/2}$ and the inclusion map $i\colon N_2 \longrightarrow M$ is (k_1+1)-connected. Assume that $k_1 \le m - k_2$ then M is homotopy equivalent to $\mathbb{CP}^{m/2}$.

Remark. We briefly note why the inequalities in the above statements are needed.
(1) Since S^{m-k_2} is compact and orientable $H^{m-k_2}(N_2;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. If $k_1+1 > m-k_2$ we know that $H^{m-k_2}(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$, but then M cannot be a cohomology sphere unless $k_2 = 0$.
(2) Since $m - k_2$ is the real dimension of N_2 $m - k_2$ must be even. If $m - k_2 = k_1 - 1$ we see that k_1 must be odd. Since the inclusion $i\colon N_2 \longrightarrow M$ is (k_1+1)-connected $H^{k_1-1}(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. But if M were to be homotopy equivalent to $\mathbb{CP}^{m/2}$ the cohomology group $H^{k_1}(M;\mathbb{Z}/p\mathbb{Z})$ would be zero. However, $e \in H^{k_1}(M;\mathbb{Z}/p\mathbb{Z})$ is then 0. Since the map $\cup e\colon H^{k_1}(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{2k_1}(M;\mathbb{Z}/p\mathbb{Z})$ is an isomorphism, this would imply

$$H^{k_1}(M;\mathbb{Z}/p\mathbb{Z}) = H^{k_1+1}(M;\mathbb{Z}/p\mathbb{Z}) = \cdots = H^{m-k_1}(M;\mathbb{Z}/p\mathbb{Z}) = 0$$

which makes it impossible for M to be homotopy equivalent to $\mathbb{CP}^{m/2}$.

If $m - k_2 \le k_1 - 2$ we have no information about $H^{k_1}(M;\mathbb{Z}/p\mathbb{Z})$. By Theorem 2.6, this makes it impossible to use the map $\cup e\colon H^{k_1}(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{2k_1}(M;\mathbb{Z}/p\mathbb{Z})$. Hence we have no information about $H^{k_1}(M;\mathbb{Z}/p\mathbb{Z}),\ldots,H^{m-k_1}(M;\mathbb{Z}/p\mathbb{Z})$.

It would be possible to obtain some information using the map $\cup e\colon H^{k_2}(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{2k_2}(M;\mathbb{Z}/p\mathbb{Z})$, where e is the Poincare dual of the image of the fundamental class of N_2 in $H_*(M;\mathbb{Z}/p\mathbb{Z})$. This would, however, require, that k_1, k_2 and m satisfy some rather complicated inequalities, which makes the proposition useless. △

Proof (Proposition 4.4): Since manifolds are also CW-complexes it is by Lemma 4.1 enough to show that M has the $\mathbb{Z}$-cohomology ring of a sphere or a complex projective space.
(1) Since the inclusion $i\colon N_2 \longrightarrow M$ is (k_1+1)-connected and N_2 is homotopy equivalent to a sphere, we see that

$$H^1(M;\mathbb{Z}/p\mathbb{Z}) = H^2(M;\mathbb{Z}/p\mathbb{Z}) = \cdots = H^{k_1}(M;\mathbb{Z}/p\mathbb{Z}) = 0$$

and by Poincare Duality, we see that

$$H^{m-k_1}(M;\mathbb{Z}/p\mathbb{Z}) = H^{m-k_1+1}(M;\mathbb{Z}/p\mathbb{Z}) = \cdots = H^{m-1}(M;\mathbb{Z}/p\mathbb{Z}) = 0.$$

As M is path connected, compact and oriented we know that $H^0(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ and $H^m(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.

Since $e \in H^{k_1}(M;\mathbb{Z}/p\mathbb{Z}) = 0$ the class e is zero. The fact that

$$\cup e\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+k_1}(M;\mathbb{Z}/p\mathbb{Z})$$

is an isomorphism and at the same time the zero map, implies clearly that

$$H^{k_1+1}(M;\mathbb{Z}/p\mathbb{Z}) = H^{k_1+2}(M;\mathbb{Z}/p\mathbb{Z}) = \cdots = H^{m-k-1}(M;\mathbb{Z}/p\mathbb{Z}) = 0,$$

which proves that M has the $\mathbb{Z}/p\mathbb{Z}$-cohomology of S^m. The Universal Coefficient Theorem imply that M has the $\mathbb{Z}$-cohomology of S^m.
(2) Recall, that since $m - k_2$ is the real dimension of N_2, $m - k_2$ is in fact even. The dimension m of M must also be even.

Since N_2 is homotopy equivalent to $\mathbb{CP}^{(m-k_2)/2}$ we know that

$$H^i(N_2;\mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & i = 0, 2, \ldots, m - k_2 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the cohomology ring of N_2 is generated by an element in $H^2(N_2;\mathbb{Z}/p\mathbb{Z})$. Since the inclusion $i\colon N_2 \longrightarrow M$ is $(k_1 + 1)$-connected we know that $H^i(M;\mathbb{Z}/p\mathbb{Z}) \cong H^i(N_2;\mathbb{Z}/p\mathbb{Z})$ for $i < k_1 + 1$. Assume now that k_1 is even (if not, replace k_1 by $k_1 - 1$). By Poincare Duality for M we know that

$$H^i(M;\mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & i = 0, 2, \ldots, k_1 \\ 0 & i = 1, 3, \ldots, k_1 - 1 \\ \mathbb{Z}/p\mathbb{Z} & i = m - k_1, m - k_1 + 2, \ldots, m \\ 0 & i = m - k_1 + 1, m - k_1 + 1, \ldots m - 1. \end{cases}$$

Since the inclusion $i\colon N_2 \longrightarrow M$ is $(k_1 + 1)$-connected and N_2 has the cohomology ring of $\mathbb{CP}^{(m-k_2)/2}$, it follows that a generator in $H^2(M;\mathbb{Z}/p\mathbb{Z})$ generates $H^i(M;\mathbb{Z}/p\mathbb{Z})$ for $i \le k_1$, i even.

We will now use Theorem 2.6 to determine $H^i(M;\mathbb{Z}/p\mathbb{Z})$ for $i = k_1+1, \ldots, m-k_1-1$. Theorem 2.4 implies that $i\colon N_1 \longrightarrow M$ is $(m - 2k_1 + 1)$-connected. From Theorem 2.6 we see that

$$\cup e\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+k_1}(M;\mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for $k_1 - 1 < i < m - 2k_1 + 1$, a surjection for $i = k_1 - 1$ and an injection for $i = m - 2k_1 + 1$.

Let $a \neq 0 \in H^2(M;\mathbb{Z}/p\mathbb{Z})$. Since a generates $H^2(M;\mathbb{Z}/p\mathbb{Z})$ we see that $a^{k_1/2} \neq 0 \in H^{k_1}(M;\mathbb{Z}/p\mathbb{Z})$. Also, since $H^{k_1}(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ we know that $a^{k_1/2} = \gamma e$, $\gamma \neq 0$. Without loss of generality we may assume that $\gamma = 1$. Consider the map

$$\cup a^{k_1/2}\colon H^{k_1}(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{2k_1}(M;\mathbb{Z}/p\mathbb{Z}).$$

This map is also given by the isomorphism $\cup e\colon H^{k_1}(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{2k_1}(M;\mathbb{Z}/p\mathbb{Z})$, since $a^{k_1/2} = e$. Hence

$$\cup a\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+2}(M;\mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for $k_1 \leq i \leq 2k_1$. By Theorem 2.6 the map $\cup e\colon H^{k_1-1}(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{2k_1-1}(M;\mathbb{Z}/p\mathbb{Z})$ is surjective. It then follows that $H^{2k_1-1}(M;\mathbb{Z}/p\mathbb{Z}) = 0$, since we know that $H^{k_1-1}(M;\mathbb{Z}/p\mathbb{Z}) = 0$. Using the isomorphism $\cup a$, we see that $H^i(M;\mathbb{Z}/p\mathbb{Z}) = 0$ for $k_1 < i < 2k_1$, i odd. Hence

$$\cup a\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+2}(M;\mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for $i \leq 2k_1$

Since the map $\cup e$ is an isomorphism for $k_1 - 1 < i < m - 2k_1 + 1$, $H^i(M;\mathbb{Z}/p\mathbb{Z})$ are periodic with period k_1 for $i \leq m - k_1$. This implies that the map

$$\cup a\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+2}(M;\mathbb{Z}/p\mathbb{Z}),$$

is an isomorphism for $i < m - k_1 - 1$. Hence we see that $H^i(M;\mathbb{Z}/p\mathbb{Z}) = 0$ for $i \leq m - k_1$ odd and $H^i(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ for $i \leq m - k_1$ even.

Using Poincare Duality one sees that the cohomology of M satisfies

$$H^i(M;\mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

Hence M has the same $\mathbb{Z}/p\mathbb{Z}$-cohomology groups as $\mathbb{CP}^{m/2}$. We must see that the cup product structure of $H^*(M;\mathbb{Z}/p\mathbb{Z}$ is that of $H^*(\mathbb{CP}^{m/2};\mathbb{Z}/p\mathbb{Z})$. As mentioned before, $\cup a\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+2}(M;\mathbb{Z}/p\mathbb{Z})$ is an isomorphism for $i < m - k_1 - 1$. By Poincare Duality we deduce that the class a generates $H^*(M;\mathbb{Z}/p\mathbb{Z})$, that is, M has the $\mathbb{Z}/p\mathbb{Z}$-cohomology ring of $\mathbb{CP}^{m/2}$. It now follows from Claim 6 in the proof of Proposition 4.2 that M also has the $\mathbb{Z}$-cohomology ring of $\mathbb{CP}^{m/2}$. □

Theorem 4.5. *Suppose M^m is simply connected, oriented and has positive sectional curvature.*

(1) If m is odd and M contains one embedded totally geodesic compact oriented submanifold N of codimension 2 then M is homotopy equivalent to a sphere (and hence homeomorphic if $m > 3$).

(2) If $m \geq 4$ is even and M contains a compact embedded totally geodesic oriented submanifold N of codimension 2 and a compact embedded totally geodesic oriented submanifold N' of codimension $< m/2$ with $N' \cap N$ transverse, then M is homotopy equivalent to a sphere or to a complex projective space.

(3) *Suppose* $m \equiv 0 \pmod 4$ *or* $m \equiv 1 \pmod 4$ *and* $m \geq 13$. *If* M *contains a compact embedded totally geodesic oriented submanifold* N_1 *of codimension* 4 *and a compact embedded totally geodesic oriented submanifold* N_2 *of codimension* $\leq m/2 - 3$ *such that* N_1 *and* N_2 *intersect transversely, then* M *has the cohomology ring of either* S^m, $\mathbb{CP}^{m/2}$ *or* $\mathbb{HP}^{m/4}$.

Remark (Ad. 3). By Lemma 4.1 we note that if M has the $\mathbb{Z}$-cohomology ring of either S^m or $\mathbb{CP}^m$, M is in fact homotopy equivalent to S^m or $\mathbb{CP}^m$. △

Proof: We will as above use $\mathbb{Z}/p\mathbb{Z}$-coefficients and prove that the spaces have the same $\mathbb{Z}/p\mathbb{Z}$-cohomology rings. We will then refer to Claim 6 in the proof of Proposition 4.2 for the argument that the spaces have the same $\mathbb{Z}$-cohomology rings.
(1) Theorem 2.4 implies that the inclusion $i\colon N \longrightarrow M$ is $(m-3)$-connected. Hence the lower bound l in Theorem 2.6 is 1.

We note that $2 \mid m-1$ since m is odd and as M is simply connected, $H^1(M;\mathbb{Z}/p\mathbb{Z}) = 0$. Since $\cup e\colon H^1(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^3(M,\mathbb{Z}/p\mathbb{Z})$ is surjective $H^3(M;\mathbb{Z}/p\mathbb{Z}) = 0$. By Poincare Duality $H^{m-1}(M,\mathbb{Z}/p\mathbb{Z}) = 0$ and $H^{m-3}(M;\mathbb{Z}/p\mathbb{Z}) = 0$. By Theorem 2.6, this implies that

$$H^2(M;\mathbb{Z}/p\mathbb{Z}) = H^4(M;\mathbb{Z}/p\mathbb{Z}) = \cdots = H^{m-3}(M;\mathbb{Z}/p\mathbb{Z}) = 0,$$
$$H^3(M;\mathbb{Z}/p\mathbb{Z}) = H^5(M;\mathbb{Z}/p\mathbb{Z}) = \cdots = H^{m-2}(M;\mathbb{Z}/p\mathbb{Z}) = 0.$$

Hence we see that M is a $\mathbb{Z}/p\mathbb{Z}$-cohomology sphere. By the Universal Coeficient Theorem M is then a $\mathbb{Z}$-cohomology sphere. Lemma 4.1 then implies that M is homotopy equivalent to a sphere. If $\dim(M) \neq 3$ the generalized Poincare Conjecture, see [Sm] and [Fr], implies that M is homeomorphic to a sphere. In dimension 3 we can only conclude that M is homotopy equivalent to a sphere.
(2) Theorem 2.4 implies that the inclusion $i\colon N \cap N' \longrightarrow N'$ is $(m - \mathrm{codim}(N) - \mathrm{codim}(N'))$-connected. Since N and N' intersect transversely $m - \mathrm{codim}(N) - \mathrm{codim}(N') = \dim(N \cap N')$.

We note that the codimension of $N' \cap N$ in N' is also 2. This follows from the following calculation, since the fibre dimension of the corresponding normal bundles are equal. We first note that $\dim(\nu(N')) = m - \dim(N')$. Since N and N' intersect transversely $\dim(N \cap N') = m - (m - \dim(N)) - (m - \dim(N'))$. Hence

$$\begin{aligned}\dim(\nu(N \cap N')) &= \dim(N') - \dim(N \cap N')\\ &= \dim(N') - m - (m - \dim(N)) - (m - \dim(N'))\\ &= m - \dim(N)\\ &= \dim(\nu(N)).\end{aligned}$$

Since the fibre dimension of $\nu(N)$ in M is 2 the fibre dimension of $\nu(N \cap N)$ in N' is also 2.

Let $e \in H^2(M;\mathbb{Z}/p\mathbb{Z})$ be the Poincare dual of the image of the fundamental class of N in $H_{m-2}(M;\mathbb{Z}/p\mathbb{Z})$. Since the pull-back $i^*(e) \in H^2(N;\mathbb{Z}/p\mathbb{Z})$ is the Euler class of the normal bundle $\nu(N)$ in M, the pull-back of e to $H^2(N \cap N';\mathbb{Z}/p\mathbb{Z})$ is the Euler class of $\nu(N \cap N')$ in N'.

Since $N \cap N' \subseteq N'$ is also oriented, an argument similar to that given in the proof of Theorem 2.6 shows that e defines a period in the cohomology ring of N', i.e. the map

$$\cup e\colon H^i(N';\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+2}(N';\mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for certain is. Furthermore, we also get the same bounds on i in terms of l as in Theorem 2.6. Since $i\colon N \cap N' \longrightarrow N'$ is $\dim(N \cap N')$-connected the number l in Theorem 2.6 is zero.

Since the map $\cup e\colon H^0(N';\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^2(N';\mathbb{Z}/p\mathbb{Z})$ is surjective and N' is path connected $H^2(N';\mathbb{Z}/p\mathbb{Z})$ is either 0 or $\mathbb{Z}/p\mathbb{Z}$. Since $\operatorname{codim}(N') < m/2$ and $m \geq 4$ it follows from Theorem 2.4 that the inclusion is at least 3-connected. Hence $H^2(M;\mathbb{Z}/p\mathbb{Z})$ is either 0 or $\mathbb{Z}/p\mathbb{Z}$.

Consider now the codimension 2 submanifold N. Theorem 2.6 implies that

$$\cup e\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+2}(M;\mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for $1 < i < m-3$.

We consider the two cases $H^2(M;\mathbb{Z}/p\mathbb{Z}) = 0$ and $H^2(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. First we see that if $H^2(M;\mathbb{Z}/p\mathbb{Z}) = 0$ then Theorem 2.6 and Poincare Duality implies that M is a $\mathbb{Z}/p\mathbb{Z}$-cohomology sphere. Hence M is also an integral cohomology sphere and Lemma 4.1 then implies that M is homotopy equivalent to a sphere.

If $H^2(M;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$, Poincare Duality and the map $\cup e$ implies that

$$H^i(M;\mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & i = 0,2,\ldots,m, \\ 0 & i = 1,3,\ldots m-1. \end{cases}$$

Poincare Duality and the fact that the map $\cup e$ is an isomorphism for $1 < i < m-3$ implies that the cohomology ring of M is generated by one element, namely $e \in H^2(M;\mathbb{Z}/p\mathbb{Z})$. Hence M has the $\mathbb{Z}/p\mathbb{Z}$-cohomology ring of $\mathbb{CP}^{m/2}$. By Claim 6 we again conclude that M has the integral cohomology ring of $\mathbb{CP}^{m/2}$. Lemma 4.1 implies that M is homotopy equivalent to a complex projective space.

(3) Assume first that $m \equiv 0 \pmod 4$. It follows from Theorem 2.4 that the inclusion $i\colon N_2 \longrightarrow M$ is 7-connected since $\operatorname{codim}(N_2) \leq m/2 - 3$ and $m \geq 13$. From Theorem 2.4 it also follows that $i\colon N_1 \cap N_2 \longrightarrow N_2$ is $\dim(N_1 \cap N_2)$-connected, since the intersection is transverse.

We note that since N_1 intersects N_2 transversely the codimension of $N_1 \cap N_2 \subseteq N_2$ is 4.

As in the proof of (2), e defines a period in the cohomology ring of N_2 i.e. the map $\cup e\colon H^i(N_2;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+4}(N_2;\mathbb{Z}/p\mathbb{Z})$ is an isomorphism for certain is. Since $i\colon N_1 \cap$

$N_2 \longrightarrow N_2$ is $\dim(N_1 \cap N_2)$-connected the number l in Theorem 2.6 is zero. Hence

$$\cup e \colon H^i(N_2; \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+4}(N_2; \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for $0 < i < \dim(N_1 \cap N_2)$, a surjection for $i = 0$ and an injection for $i = \dim(N_1 \cap N_2)$.

As $i \colon N_2 \longrightarrow M$ is 7-connected, we know that $H^i(N_2; \mathbb{Z}/p\mathbb{Z}) \cong H^i(M; \mathbb{Z}/p\mathbb{Z})$ for $i < 7$, hence we deduce that

$$\cup e \colon H^i(M; \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+4}(M; \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for $0 < i < 7$ and a surjection for $i = 0$.

The map $i \colon N_1 \longrightarrow M$ is $(m-7)$-connected, hence the number l in Theorem 2.6 is 3. Thus we see that

$$\cup e \colon H^i(M; \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+4}(M; \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for $3 < i < m-7$, a surjection for $i = 3$ and an injection for $i = m-7$. Combining these two facts yields that

$$\cup e \colon H^i(M; \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+4}(M; \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for $0 < i < m-7$, a surjection for $i = 0$ and an injection for $i = m-7$.

We consider the two cases $H^4(M; \mathbb{Z}/p\mathbb{Z}) = 0$ and $H^4(M; \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. We see that if $H^4(M; \mathbb{Z}/p\mathbb{Z}) = 0$ then $e = 0$. Since $\cup e$ is also an isomorphism (injection) we deduce that

$$H^1(M; \mathbb{Z}/p\mathbb{Z}) = \cdots = H^{m-3}(M; \mathbb{Z}/p\mathbb{Z}) = 0.$$

By Poincare Duality, we see that M is a $\mathbb{Z}/p\mathbb{Z}$-cohomology (and hence integral cohomology) sphere.

If $H^4(M; \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ and $H^2(M; \mathbb{Z}/p\mathbb{Z}) = 0$, we use the isomorphism $\cup e$ and Poincare Duality to deduce that M has the $\mathbb{Z}/p\mathbb{Z}$-cohomology ring of $\mathbb{HP}^{m/4}$. By Claim 6 we deduce that M has the integral cohomology ring of $\mathbb{HP}^{m/4}$.

If $H^4(M; \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ and $H^2(M; \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ we use Poincare Duality to deduce that $e = x \cup y$. As in the proof of Proposition 4.2 we see that the map

$$\cup x \colon H^i(M; \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+2}(M; \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for $0 < i < m-7$. By Poincare Duality, we deduce that M has the $\mathbb{Z}/p\mathbb{Z}$-cohomology of $\mathbb{CP}^{m/2}$. Hence, by Claim 6, M has the same integral cohomology ring as $\mathbb{CP}^{m/2}$.

Assume now that $m \equiv 1 \pmod 4$. Since M is simply connected $H^1(M; \mathbb{Z}/p\mathbb{Z}) = 0$. As the argument that $\cup e$ is an isomorphism for $0 < i < m-7$ did not depend on whether

$m \equiv 1 \pmod 4$ or $m \equiv 0 \pmod 4$ we know that $\cup e\colon H^i(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{i+4}(M;\mathbb{Z}/p\mathbb{Z})$ is an isomorphism for $0 < i < m-7$, injective for $i = m-7$ and surjective for $i = 0$.

By applying the isomorphism $\cup e$ we see that

$$H^1(M;\mathbb{Z}/p\mathbb{Z}) = H^5(M;\mathbb{Z}/p\mathbb{Z}) = \cdots = H^{m-9}(M;\mathbb{Z}/p\mathbb{Z}) = 0.$$

By Poincare Duality we see that $H^9(M;\mathbb{Z}/p\mathbb{Z}) = 0$. As $m \equiv 1 \pmod 4$ we see that $4 \mid m-9$. Since

$$\cup e^{(m-9)/4}\colon H^4(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{m-9}(M;\mathbb{Z}/p\mathbb{Z})$$

is an isomorphism we deduce that $H^4(M;\mathbb{Z}/p\mathbb{Z}) = 0$ and hence $e = 0$. Hence

$$H^1(M;\mathbb{Z}/p\mathbb{Z}) = H^2(M;\mathbb{Z}/p\mathbb{Z}) = \cdots = H^{m-4}(M;\mathbb{Z}/p\mathbb{Z}) = 0.$$

and by Poincare Duality we see that M is a $\mathbb{Z}/p\mathbb{Z}$-cohomology (and hence integral cohomology) sphere. □

Remark (Ad. 2). As in [W I] one could try to apply Proposition 4.2 to the pair $(N', N \cap N')$ and conclude that N' has the homotopy type of either S^n or $\mathbb{CP}^{m-k'}$, where k' is the codimension of N' in M. One could then apply Theorem 2.4, Theorem 2.6 and Lemma 4.1 to conclude that M has the same homotopy type as N'. There is however no way of knowing that $\dim(N \cap N')$ divides $\dim(N')$ (not even if the intersection is transverse) which is an essential assumption in the proof of Proposition 4.2. △

The next theorem asserts that the cohomology groups $H^i(M;\mathbb{Z})$ $i = 1, \ldots, m-1$ consists only of 2-torsion, if certain conditions are satisfied.

Theorem 4.6. *Assume that M^m has positive sectional curvature. Suppose $N_1^{m-k_1}$ and $N_2^{m-k_2}$ are two totally geodesic submanifolds intersecting transversely. Furthermore, assume that $k_1 \le (m+1)/3$, $k_1 + 2k_2 \le m$ and that k_1 is odd. Then for all $i \in \{1, \ldots, m-1\}$ and $x \in H^i(M;\mathbb{Z})$ we have $2x = 0$.*

Proof: Theorem 2.4 implies that the inclusion $i\colon N_1 \longrightarrow M$ is $(m-2k_1+1)$-connected. By Theorem 2.6 there is a class $e \in H^{k_1}(M;\mathbb{Z})$ such that $i^*(e) \in H^{k_1}(N_1;\mathbb{Z})$ is the Euler class of the normal bundle $\nu(N_1)$ in M and such that the map $\cup e\colon H^i(M;\mathbb{Z}) \longrightarrow H^i(M;\mathbb{Z})$ is an isomorphism for $i = k_1, \ldots, m-k_1$. Since the codimension k_1 is odd Proposition 1.19 implies that $2i^*(e) = 0$.

Since $k_1 < (m+1)/3$ the inclusion $i\colon N_1 \longrightarrow M$ is k_1-connected which implies that the map $H^{k_1}(N_1;\mathbb{Z}) \longrightarrow H^{k_1}(M;\mathbb{Z})$ is injective. Hence $i^*(2e) = 2i^*(e) = 0$ implies that $2e = 0$. Since the map $\cup e$ is an isomorphism for $i = k_1, \ldots, m-k_1$ this proves the assertion for $i = k_1, \ldots, m-k_1$.

Assume that $k_1 \leq k_2$. By Theorem 2.4 the inclusion $N_1 \cap N_2 \longrightarrow N_2$ is $(m - k_1 - k_2)$-connected. As in the proof of Theorem 4.5 (2) we see that the pull-back $i^*(e) \in H^{k_1}(N_1 \cap N_2; \mathbb{Z})$ is the Euler class of the normal bundle $\nu(N_1 \cap N_2)$ in N_2. Since $i\colon N_1 \cap N_2 \longrightarrow N_2$ is $(m-k_1-k_2)$-connected, we deduce from Theorem 2.6 that $\cup e\colon H^i(N_2; \mathbb{Z}) \longrightarrow H^{k_1+i}(N_2; \mathbb{Z})$ is an isomorphism for $1 \leq i \leq m - k_2 - k_1 - 1$.

We note that the fibre dimension of $\nu(N_1 \cap N_2)$ in N_2 is odd. This follows from the calculation in Theorem 4.5 (2) since N_1 and N_2 intersect transversely and the fibre dimension of $\nu(N_1)$ in M is odd. By Proposition 1.19, this implies that $2i^*(e) \in H^{k_1}(N_1 \cap N_2; \mathbb{Z})$ is zero. Since the inclusion $i\colon N_1 \cap N_2 \longrightarrow N_2$ is k_1-connected we see that $i^*(2e) = 2i^*(e) = 0$ implies that $2e = 0$ in $H^{k_1}(N_2; \mathbb{Z})$. Hence $2 \cdot H^i(N_2; \mathbb{Z}) = 0$ for $1 \leq i \leq m - k_2 - 1$ since $\cup e$ is an isomorphism for $1 \leq i \leq m - k_2 - k_1 - 1$.

We note that $m - k_2 - 1 \geq k_1$ since we assumed $m - 2k_2 \geq k_1$. Theorem 2.4 implies that the inclusion $i\colon N_2 \longrightarrow M$ is $(m - 2k_2 + 1)$-connected and hence k_1-connected. Hence $2 \cdot H^i(N_2; \mathbb{Z}) = 0$ for $1 \leq i \leq m - k_2 - 1$ implies that $2 \cdot H^i(M; \mathbb{Z}) = 0$ for $i = 1, \cdots, k_1$. We now know that $2 \cdot H^i(M; \mathbb{Z}) = 0$ for $1 \leq i \leq m - k_1$. By Poincare Duality $2 \cdot H^i(M; \mathbb{Z}) = 0$ for $1 \leq i \leq m - 1$.

If $k_1 > k_2$ then we consider the inclusion $N_1 \cap N_2 \longrightarrow N_1$ which is $(m - k_1 - k_2)$-connected. By the above argument we see that $H^*(N_1; \mathbb{Z})$ has k_2 as a period, that is, the map $\cup e\colon H^i(N_1; \mathbb{Z}) \longrightarrow H^{i+k_2}(N_1; \mathbb{Z})$ is an isomorphism for $i = 1, \ldots, m - k_1 - 1$. Since we have already established that $2 \cdot H^i(M; \mathbb{Z}) = 0$ for $i = k_1, \ldots, m - k_1$ and the inclusion $N_1 \longrightarrow M$ is $m - 2k_1 + 1$-connected, it follows that $2 \cdot H^i(M; \mathbb{Z}) = 0$ for $1 \leq i \leq m - k_1$. We finish the proof by applying Poincare Duality. □

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